

Dynamic Modelling and Analysis of Robotic Manipulators with Mixed Rigid-Elastic Joints

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ABSTRACT Rigid joints are popular in traditional robotic manipulator designs due to their fast-response ability and high position control precision, while elastic joints are well known for their safety. Robotic manipulators using both rigid and elastic joints are developed to harness the benefits and mitigate the drawbacks of these joint types. The integration of mixed rigid-elastic joints in robotic manipulators offers significant potential for enhancing adaptability, precision, and resilience in dynamic and unstructured environments. This approach is particularly advantageous in applications that demand a balance between strength and compliance. This paper presents a systematic, tutorial-like approach to modelling and analysis of robotic manipulators with mixed rigid-elastic joints. Firstly, the dynamic equations are developed using the Lagrangian method. In this paper, the constraint on motor positions is relaxed, which means the motors can be placed on any link on the robotic manipulator. In addition, properties of the mathematical model, including skew-symmetry and linearity with respect to dynamic parameters, are presented. The representation of the dynamic matrices and vectors with respect to the dynamic parameters is introduced. Secondly, the recursive Newton-Euler algorithm is adopted to derive the model of robots with mixed rigid-elastic joints. An inverse dynamic algorithm is presented to calculate the forces/torques from the provided accelerations, velocities, and positions of the joints. Then, a forward dynamic algorithm is proposed to calculate the accelerations, velocities, and positions of the joints from the provided forces/torques. In addition, the impact of varying joint stiffness and the placement of elastic joints is analysed. Finally, an example and simulation are provided to demonstrate the modelling process of robotic manipulators with mixed rigid-elastic joints.

INDEX TERMS robotic manipulator with mixed rigid-elastic joints, recursive Newton-Euler, Lagrange, modelling

I. INTRODUCTION

In industrial applications, rigid joints are widely used due to their high positional accuracy and rapid response to changes in desired positions. However, robots with rigid joints may exert excessive force when interacting with their surroundings, posing a risk of damaging objects or surfaces during contact. On the other hand, elastic joints are well-known due to their low interaction force. Therefore, they can be considered for improving safety. Additionally, elastic joints can function as shock absorbers, protecting the robot's joints from undesirable impacts during interactions. However, they may introduce oscillations, which can reduce positional accuracy.

To leverage the advantages of both rigid and elastic joints, robotic manipulators with mixed rigid-elastic joints are introduced. In these robotic manipulators, some joints are de-

signed to be elastic, while others remain rigid. The elasticity in the joints can be introduced through elastic elements, such as springs [1], [2] or custom-designed components [3], [4]. Alternatively, it can result from the properties of motion transmission mechanisms, including harmonic drives, belts, or long shafts [5].

Although previous work has introduced this concept, a comprehensive and systematic resource for the mathematical modelling of robotic manipulators with mixed rigid-elastic joints is, to the best of the authors' knowledge, still lacking. Therefore, this paper aims to offer a clear, accessible, comprehensive, and systematic approach to the modelling process using different methods of robotic manipulators with mixed joint types.

In our previous works [6], [7], the mathematical model

of robotic manipulators with mixed rigid-elastic joints was derived using the Lagrangian method and Newton-Euler method, respectively. In these works, the assumption about the positions of motors is relaxed, which means that the motor can be placed anywhere on the robot. This makes the dynamic model more realistic and extensible. In addition, by introducing the unified rigid body definition, consisting of a link and all motors attached to that link, the positions of the centres of mass of the link and all attached motors can be merged into the position of the centre of mass of the unified rigid body. The main advantage of the modelling method using the Newton-Euler method in this work is the use of the unified rigid body definition, which helps simplify the calculation process. In addition, while high-order derivative was used in [8], only accelerations of the origins of frames are required in the proposed algorithm. This helps reduce numerical error during the calculation process.

In this paper, the work is extended. In particular, the skew-symmetric property is analysed for the mathematical model of this type of robotic manipulator. The linearity of the dynamic model with respect to the dynamic parameters is also presented. In addition, the representation of the dynamic matrices, such as the mass matrix, Coriolis and centrifugal matrix, and gravitational vector, is derived for this type of robotic manipulator. Furthermore, the effect of different joint stiffness and different elastic joint positions on the manipulator's behaviour is analysed. An example is also provided. Simulations are carried out to verify the proposed algorithm. The important takeaways are summarised in Appendix A. Readers may find it helpful to begin with this section, as it provides an overview of the notable results before delving into the detailed derivations presented in each section. In addition, a dynamics calculation program is provided as supplementary material and described in Appendix B to illustrate and clarify the calculation steps.

This paper is organised as follows. Related works are presented in Section II. In Section III, the model assumptions, a brief description of the kinematics, and notations are introduced. The mathematical models of robotic manipulators with all elastic joints and the generalisation to robotic manipulators with mixed rigid-elastic joints are derived using the Lagrangian method in Sections IV and V, respectively. The skew-symmetric property is discussed in Section VI, while the linearity of the dynamic model with respect to the dynamic parameters is discussed in Section VII. Then, in Section VIII, the dynamic matrices are represented in terms of the dynamic parameters. The distinction between Sections VII and VIII lies in their focus: Section VII examines the linear representation of the entire dynamic equation in terms of dynamic parameters, while Section VIII focuses on the linear representation of each dynamic matrix individually. In Section IX, the modelling process for robotic manipulators with mixed rigid-elastic joints using the recursive Newton-Euler algorithm is presented. Based on this algorithm, a procedure to calculate the forward dynamics is proposed in Section X. The effect of different stiffness on robots with mixed rigid-

elastic joints is studied on a robotic system in Section XI. The effect of different positions of the elastic joint is discussed in Section XII. Design heuristics are discussed in Section XIII. Simulations are conducted based on an example model and the results are presented in Sections XIV and XV. Finally, the conclusions are drawn in Section XVI. The important results are summarised in Appendix A. A dynamics calculation program is provided as supplementary material and described in Appendix B.

II. RELATED WORKS

One of the earliest studies related to elastic joint robots is [9], where a simplified model is introduced. Series elastic actuator (SEA), proposed in [1], [10], is one of the notable designs adopting the elastic joint mechanism. In [11], a biomimetic walking robot utilising SEA is presented. In [12], various SEA configurations are discussed, and their dynamic behaviours are analysed. Elastic joints are used in various applications, including robotic manipulators [9], [13], humanoid robots [3], [14]–[16], and legged robots [17]. Many control algorithms are also introduced for robots with elastic joints [5], [18]–[22].

The dynamics of robots with elastic joints are more complex than those with purely rigid joints due to the coupling between the motor and joint in elastic configurations. The modelling of robots with elastic joints with nonlinear damping and hysteresis is presented in [23]. A model for robots with mixed rigid-elastic joints and an inverse dynamic control algorithm are introduced in [6]. In [24], robots with variable stiffness actuators are modelled using the Koopman operator-based modelling method.

Using the Lagrangian method, a complete mathematical model of robots with all elastic joints is developed in [25]. This dynamic model considers the coupling dynamics between the joints and the motors. In some cases, the kinetic energy of the motors is assumed to be due to their own rotation [9]. As a result, reduced dynamic models are considered [9], where the coupling terms are ignored. Besides the Lagrangian method, the Newton-Euler recursive method is considered to model robots with all elastic joints in [8]. The mathematical model of robots with mixed rigid-elastic joints is similar to that of robots with all elastic joints, except that the dimension of the elastic joint dynamics is decreased. The mathematical model of this type of robotic manipulator, derived using the Lagrangian method, is introduced in [5], [6], [26]. Furthermore, the Newton-Euler method is utilised for deriving the mathematical model of robots with mixed rigid-elastic joints in [7], and in other systems such as an underwater robot [27].

The Newton-Euler recursive method is adopted to derive the dynamic model of robots with all elastic joints in [8]. The work [28] presents a completed dynamic model of the dual-arm system called the Dual-Arm Elastic Joints Newton-Euler Algorithm (DA-EJNEA). In [29], symbolic differentiation of the recursive Newton-Euler algorithm (SDRNEA) is proposed to automatically generate the inverse dynamics of

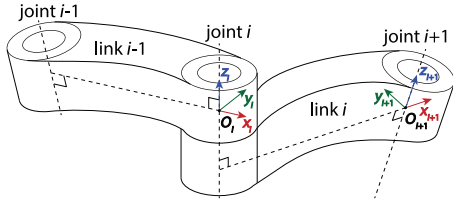


FIGURE 1: The kinematic diagram of link $i - 1$ and link i .

serial robots with flexible joints in symbolic form. A Newton-Euler approach is used to model a flexibly jointed manipulator in [30], which is then applied to analyse the control problem. Any function representing the coupling between the motor and the link can be incorporated into the simulation model, including torsional spring models, nonlinear springs, backlash, and hysteresis. A method for the symbolic, recursive computation of the inverse dynamics of elastic joint robots, accounting for gyroscopic effects introduced by the drive-trains, is proposed in [31]. Higher-order time derivatives of spatial operators are used to improve the computational effort and preserve the structure of the system's equations.

III. PRELIMINARIES AND NOTATIONS

A. PRELIMINARIES

In this paper, scalar variables are shown in *italic*, not-bold symbols (e.g. a, A), vectors are described in not-italicised, bold symbols (e.g. $\mathbf{a}, \mathbf{A}$), and matrices are written in *italic*, bold symbols (e.g. $\mathbf{a}, \mathbf{A}$).

A fully actuated robotic manipulator of interest in this paper is assumed to be an open kinematic chain, where $N + 1$ rigid bodies (the base link 0 and N links) are interconnected by N joints and driven by N electrical motors. Link $i - 1$ and link i are connected by joint i as illustrated in Figure 1. A coordinate system i (frame i) is assigned to joint i , where the axis z_i is located along the axis of rotation if joint i is a revolute joint, and along the translational axis if joint i is a prismatic joint. Then, the axes x_i and y_i are determined following the Denavit-Hartenberg convention. As shown in Figure 1, frame i is rigidly attached to link $i - 1$. This implies that all points on link $i - 1$ have constant coordinates in frame i , which will be useful later.

In this research, the following assumptions are made:

A1: The elasticity region is limited to the linear elasticity region, where Hooke's law is applicable.

A2: The motors are modelled as uniform bodies with their centres of mass on the rotation axis.

A3: Only rotary actuators are considered in this work.

Every linear spring has a range where Hooke's law is applicable, which is illustrated from the origin until point P in Figure 2. When the spring is deformed beyond this region, it will be damaged and undergo permanent deformation. The assumption **A1** implies that the deviation between the angular positions of the motor and the joint of the elastic joints in this work will be limited within this region. In addition, most of the motors are approximately symmetric around their shaft.

As a result, by making assumption **A2**, the inertia tensor of motors can be simplified. Furthermore, the modelling processes of robotic manipulators equipped with rotary or linear actuators are mathematically analogous. Therefore, assumption **A3** is made to avoid redundant derivations and maintain clarity of presentation.

In previous studies [5], [26], motors are assumed to be placed on one of the bodies preceding the driven link, e.g. motor $i + 1$, which drives joint $i + 1$, is considered to be attached to link i . However, in some robotic manipulator designs, the motors are placed as close as possible to the base link mounted to the ground. This helps decrease the total inertia of the robotic manipulator, reduces the load for the joints close to the base, and moves the centre of mass of the robotic manipulator close to the base link, improving the stability of the entire robotic manipulator. On the other hand, in some other designs, motors are placed as close as possible to the driven links. This is to decrease the mechanical transmission length, reducing friction, backlashes, and fatigue in transmission mechanisms. Therefore, in this work, the assumption of motor position is loosened, which means that motors can be placed on any link on the robotic manipulator.

Let l_{m_i} denote the link on which motor i is located. Figures 3a and 3b show the arrangement motor i , attached to link l_{m_i} , for rigid joints and elastic joints, respectively. With each motor i , a coordinate system (frame M_i) is defined as attached to motor i , and its z -axis is located along the motor's rotation axis. Then, on the assumption **A2**, the inertia tensor of motor i expressed in frame M_i is a constant diagonal matrix. For rigid joints, the torque from motor i is transmitted directly to joint i through a gearbox with transmission ratio η_{m_i} . For elastic joint i , the output of motor i is transmitted to the elastic coupling through a gearbox with transmission ratio η_{m_i} and the output of the elastic coupling is transmitted to joint i via a torque transmission mechanism with transmission ratio η_{l_i} . As noted in [33], if a gearbox or torque transmission mechanism changes the type of motion (from rotational to linear motion or vice versa), the transmission ratios are dimensional quantities, i.e. a quantity with a physical unit (for example $^\circ/\text{m}$, cm/rad , etc). On the other hand, if a gearbox or torque transmission mechanism does not change the type of motion, the transmission ratios are dimensionless quantities, and in most cases, they are greater than 1.

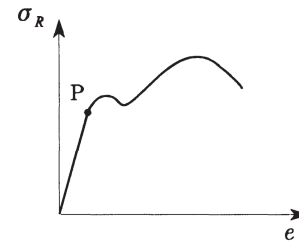


FIGURE 2: Load-deformation graph of a spring [32]. σ_R denotes the load effort and e denotes the deformation of a spring.

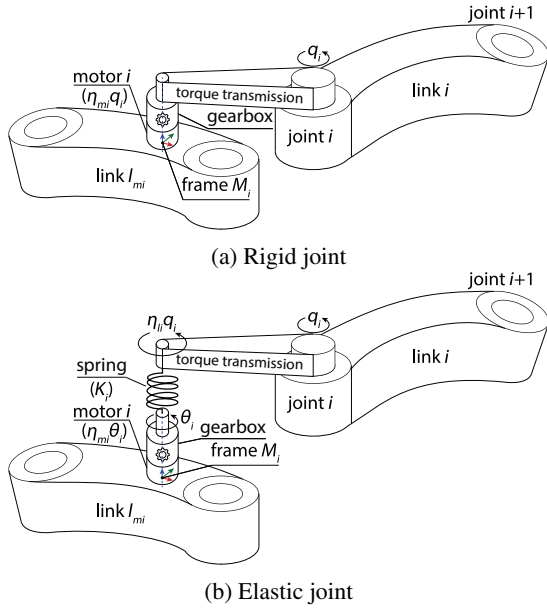


FIGURE 3: The kinematic diagram representing elastic joints and rigid joints.

B. NOTATIONS

The following notations are used for link and motor parameters, and dynamic matrices in this paper:

$\mathbf{q}$	vector of joint variables
$\boldsymbol{\theta}$	vector of positions of motors of elastic joints
η_{li}	transmission ratio at the output of joint i (if joint i is an elastic joint)
η_{mi}	gear ratio of motor i
$\mathbf{I}_{li}^{i+1}$	inertia tensor of link i , expressed in frame $i+1$
$\mathbf{I}_{mi}^{M_i}$	inertia tensor of motor i , expressed in frame M_i
m_{li}	mass of link i
m_{mi}	mass of motor i
$\mathbf{P}_{li}^0$	position vector of the centres of mass of link i expressed in the base frame 0
$\mathbf{P}_{mi}^0$	position vector of the centres of mass of motor i expressed in the base frame 0
$\mathbf{v}_{cli}^0$	translational velocity of the centre of mass of link i expressed in the base frame 0
$\boldsymbol{\omega}_{li}^{i+1}$	angular velocity of link i expressed in frame $i+1$
$\mathbf{v}_{cmi}^0$	translational velocity of the centre of mass of motor i expressed in the base frame 0
$\boldsymbol{\omega}_{mi}^{M_i}$	angular velocity of motor i expressed in frame M_i
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{S}$	coupling matrix
$\mathbf{B}$	inertia matrix of the motors of elastic joints
$\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2$	Coriolis and centrifugal matrices

$\mathbf{G}$	gravitational vector
$\boldsymbol{\tau}_{mr}$	vector of motor torques of the rigid joints
$\boldsymbol{\tau}_{me}$	vector of motor torques of the elastic joints
$\boldsymbol{\tau}_{fr,l}$	vector of friction torques of the joints
$\boldsymbol{\tau}_{fr,m}$	vector of friction torques of the motors of elastic joints
$\boldsymbol{\tau}_{ex}$	vector of external torques affecting the joints

IV. MODELLING ROBOTIC MANIPULATORS WITH ALL ELASTIC JOINTS USING THE LAGRANGIAN METHOD

In this section, the dynamic equations of robotic manipulators with all elastic joints are derived using the Lagrangian method. Let q_i denote the joint variable of joint i , and let θ_i denote the angular position of motor i , after the corresponding gearbox. The vector of joint variables is denoted as $\mathbf{q} = [q_1 \ q_2 \ \dots \ q_N]^T$, and the vector of motor angular positions is denoted as $\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \dots \ \theta_N]^T$.

A. POTENTIAL ENERGY

The potential energy, denoted as $\mathbb{U}(\mathbf{q}, \boldsymbol{\theta})$, consists of the gravitational potential energy, denoted as $\mathbb{U}_g(\mathbf{q})$, and the elastic potential energy, denoted as $\mathbb{U}_e(\mathbf{q}, \boldsymbol{\theta})$:

$$\mathbb{U}(\mathbf{q}, \boldsymbol{\theta}) = \mathbb{U}_g(\mathbf{q}) + \mathbb{U}_e(\mathbf{q}, \boldsymbol{\theta}) \quad (1)$$

Potential Energy due to Gravity

Let $\mathbf{P}_{mi}^0 = [P_{mx_i}^0 \ P_{my_i}^0 \ P_{mz_i}^0]^T$ and $\mathbf{P}_{li}^0 = [P_{lx_i}^0 \ P_{ly_i}^0 \ P_{lz_i}^0]^T$ denote the position vectors of the centres of mass of motor i and link i expressed in the base frame 0, respectively. Let m_{mi} and m_{li} denote the mass of motor i and the mass of link i , respectively. Let $\mathbf{g}^0 = [0 \ 0 \ -g]^T$, where $g = 9.81 \text{ m/s}^2$, denote the gravitational acceleration vector. Then, the potential energy due to gravity is calculated as follows:

$$\begin{aligned} \mathbb{U}_g(\mathbf{q}) &= \sum_{i=1}^N m_{mi} g P_{mz_i}^0 + \sum_{i=1}^N m_{li} g P_{lz_i}^0 \\ &= - \sum_{i=1}^N (m_{mi} (\mathbf{g}^0)^T \mathbf{P}_{mi}^0 + m_{li} (\mathbf{g}^0)^T \mathbf{P}_{li}^0) \end{aligned} \quad (2)$$

Potential Energy due to Joint Elasticity

Let K_i denote the stiffness coefficient of the elastic coupling at joint i . Then $\mathbf{K} = \text{diag}(K_1, K_2, \dots, K_N)$ and $\boldsymbol{\eta}_l = \text{diag}(\eta_{l_1}, \eta_{l_2}, \dots, \eta_{l_N})$ denote the matrices containing the stiffness coefficients and the torque transmission ratios, respectively. Then, the potential energy due to the joint elasticity is calculated as follows:

$$\mathbb{U}_e(\mathbf{q}, \boldsymbol{\theta}) = \frac{1}{2} (\boldsymbol{\eta}_l \mathbf{q} - \boldsymbol{\theta})^T \mathbf{K} (\boldsymbol{\eta}_l \mathbf{q} - \boldsymbol{\theta}) \quad (3)$$

B. KINETIC ENERGY

The kinetic energy, denoted as $\mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})$, consists of the kinetic energy of the centres of mass of the links, denoted as $\mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}})$, and the kinetic energy of the centres of mass of the motors, denoted as $\mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})$:

$$\mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = \mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) \quad (4)$$

Kinetic Energy of the Links

Let $\mathbf{I}_l^{i+1}$ denote the inertia tensor of link i , expressed in frame $i+1$. By expressing the inertia tensor of link i in frame $i+1$, which is rigidly attached to link i , $\mathbf{I}_l^{i+1}$ becomes a constant symmetric matrix independent of the joint configuration. Let $\mathbf{v}_{cl_i}^0$ and $\boldsymbol{\omega}_l^{i+1}$ denote the translational velocity of the centre of mass of link i , expressed in the base frame, and the angular velocity of the same link, expressed in frame $i+1$, respectively. The kinetic energy of all links is given by:

$$\mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^N (m_{l_i} (\mathbf{v}_{cl_i}^0)^T \mathbf{v}_{cl_i}^0 + (\boldsymbol{\omega}_l^{i+1})^T \mathbf{I}_l^{i+1} \boldsymbol{\omega}_l^{i+1}) \quad (5)$$

Let us introduce $3 \times N$ Jacobian matrices $\mathbf{J}_{vl_i}^0$ containing partial velocity vectors related to translational velocity $\mathbf{v}_{cl_i}^0$, and $\mathbf{J}_{\omega_l}^{i+1}$ including partial velocity vectors associated with angular velocity $\boldsymbol{\omega}_l^{i+1}$, and let $\mathbf{0}$ denote a zero vector, then:

$$\mathbf{v}_{cl_i}^0 = \mathbf{J}_{vl_i}^0 \dot{\mathbf{q}} \quad (6)$$

$$\boldsymbol{\omega}_l^{i+1} = \mathbf{J}_{\omega_l}^{i+1} \dot{\mathbf{q}} \quad (7)$$

where

$$\mathbf{J}_{vl_i}^0 = \begin{bmatrix} \frac{\partial \mathbf{P}_{l_i}^0}{\partial q_1} & \frac{\partial \mathbf{P}_{l_i}^0}{\partial q_2} & \dots & \frac{\partial \mathbf{P}_{l_i}^0}{\partial q_i} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{J}_{\omega_l}^{i+1} &= (\mathbf{R}_{i+1}^0)^T \mathbf{J}_{\omega_l}^0 \\ &= (\mathbf{R}_{i+1}^0)^T [\rho_1 \mathbf{z}_1^0 \quad \rho_2 \mathbf{z}_2^0 \quad \dots \quad \rho_i \mathbf{z}_i^0 \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \\ &= [\rho_1 \mathbf{z}_1^{i+1} \quad \rho_2 \mathbf{z}_2^{i+1} \quad \dots \quad \rho_i \mathbf{z}_i^{i+1} \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \end{aligned}$$

In the Jacobian matrix $\mathbf{J}_{\omega_l}^{i+1}$, vector $\mathbf{z}_k^{i+1}$ is the unit vector in the direction of the z -axis of the coordinate system k , expressed in frame $i+1$. The vector $\mathbf{z}_k^0$ is the third column of the rotation matrix $\mathbf{R}_k^0$. According to the Denavit-Hartenberg convention, the z -axis is the axis of rotation for a revolute joint and the axis of translation for a prismatic joint. The constant $\rho_i = 1$ if joint i is a revolute joint, and $\rho_i = 0$ if joint i is a prismatic joint. With the introduction of the Jacobian matrices, the kinetic energy of all links in the system becomes:

$$\begin{aligned} \mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} \sum_{i=1}^N (m_{l_i} \dot{\mathbf{q}}^T (\mathbf{J}_{vl_i}^0)^T \mathbf{J}_{vl_i}^0 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\mathbf{J}_{\omega_l}^{i+1})^T \mathbf{I}_l^{i+1} \mathbf{J}_{\omega_l}^{i+1} \dot{\mathbf{q}}) \\ &= \frac{1}{2} \dot{\mathbf{q}}^T \left[\sum_{i=1}^N (m_{l_i} (\mathbf{J}_{vl_i}^0)^T \mathbf{J}_{vl_i}^0 + (\mathbf{J}_{\omega_l}^{i+1})^T \mathbf{I}_l^{i+1} \mathbf{J}_{\omega_l}^{i+1}) \right] \dot{\mathbf{q}} \\ &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_l(\mathbf{q}) \dot{\mathbf{q}} \end{aligned} \quad (8)$$

where

$$\mathbf{M}_l(\mathbf{q}) = \sum_{i=1}^N (m_{l_i} (\mathbf{J}_{vl_i}^0)^T \mathbf{J}_{vl_i}^0 + (\mathbf{J}_{\omega_l}^{i+1})^T \mathbf{I}_l^{i+1} \mathbf{J}_{\omega_l}^{i+1}) \quad (9)$$

is the $N \times N$ inertia matrix of the links.

Kinetic Energy of the Motors

Let $\mathbf{I}_{m_i}^{M_i}$ denote the inertia tensor of motor i , expressed in frame M_i . According to assumption **A2**, $\mathbf{I}_{m_i}^{M_i}$ is a constant diagonal matrix, i.e. $\mathbf{I}_{m_i}^{M_i} = \text{diag}(I_{mx_i}, I_{my_i}, I_{mz_i})$, where I_{mx_i} , I_{my_i} , and I_{mz_i} are constants. Let $\mathbf{v}_{cm_i}^0$ and $\boldsymbol{\omega}_{m_i}^{M_i}$ denote the translational velocity of the centre of mass of motor i , expressed in the base frame, and the angular velocity of motor i , expressed in frame M_i , respectively. The kinetic energy of the motors is given by:

$$\mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = \frac{1}{2} \sum_{i=1}^N (m_{m_i} (\mathbf{v}_{cm_i}^0)^T \mathbf{v}_{cm_i}^0 + (\boldsymbol{\omega}_{m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \boldsymbol{\omega}_{m_i}^{M_i}) \quad (10)$$

We introduce the $3 \times N$ Jacobian matrices $\mathbf{J}_{vm_i}^0$ for the translational velocity $\mathbf{v}_{cm_i}^0$, expressed in the base frame 0, and $\mathbf{J}_{\omega_{m_i}}^{M_i}$ for the angular velocity $\boldsymbol{\omega}_{m_i}^{M_i}$, expressed in frame M_i . Let $\mathbf{z}_{m_i}^0$ denote the unit vector corresponding to the axis of rotation of motor i expressed in the base frame 0. In addition, when being expressed in frame M_i , $\mathbf{z}_{m_i}^0$ becomes $\mathbf{z}_{m_i}^{M_i} = \mathbf{z}_0 = [0 \ 0 \ 1]^T$. Note that, the linear velocity is only dependent on joint variables. In addition, the angular velocity of the motor i is a vectorial summation of the angular velocity vector of link l_{m_i} and the angular rotation of the motor about its own axis of rotation. Then, with the assumption **A3**, we have:

$$\mathbf{v}_{cm_i}^0 = \mathbf{J}_{vm_i}^0 \dot{\mathbf{q}} \quad (11)$$

$$\boldsymbol{\omega}_{m_i}^{M_i} = \mathbf{J}_{\omega_{m_i}}^{M_i} \dot{\mathbf{q}} + \mathbf{z}_{m_i}^{M_i} \eta_{m_i} \dot{\theta}_i = \mathbf{J}_{\omega_{m_i}}^{M_i} \dot{\mathbf{q}} + \mathbf{z}_0 \eta_{m_i} \dot{\theta}_i \quad (12)$$

where

$$\mathbf{J}_{vm_i}^0 = \begin{bmatrix} \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_1} & \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_2} & \dots & \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_{l_{m_i}}} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad (13)$$

and

$$\begin{aligned} \mathbf{J}_{\omega_{m_i}}^{M_i} &= (\mathbf{R}_{M_i}^0)^T \mathbf{J}_{\omega_{m_i}}^0 \\ &= (\mathbf{R}_{M_i}^0)^T [\rho_1 \mathbf{z}_1^0 \quad \rho_2 \mathbf{z}_2^0 \quad \dots \quad \rho_{l_{m_i}} \mathbf{z}_{l_{m_i}}^0 \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \\ &= [\rho_1 \mathbf{z}_1^{M_i} \quad \rho_2 \mathbf{z}_2^{M_i} \quad \dots \quad \rho_{l_{m_i}} \mathbf{z}_{l_{m_i}}^{M_i} \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \end{aligned} \quad (14)$$

With the introduction of the Jacobian matrices, the kinetic energy of all motors becomes:

$$\begin{aligned} \mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) &= \frac{1}{2} \sum_{i=1}^N (m_{m_i} \dot{\mathbf{q}}^T (\mathbf{J}_{vm_i}^0)^T \mathbf{J}_{vm_i}^0 \dot{\mathbf{q}} \\ &\quad + (\dot{\mathbf{q}}^T (\mathbf{J}_{\omega_{m_i}}^{M_i})^T + (\mathbf{z}_0)^T \eta_{m_i} \dot{\theta}_i) \mathbf{I}_{m_i}^{M_i} (\mathbf{J}_{\omega_{m_i}}^{M_i} \dot{\mathbf{q}} + \mathbf{z}_0 \eta_{m_i} \dot{\theta}_i)) \\ &= \frac{1}{2} \sum_{i=1}^N (\dot{\mathbf{q}}^T (m_{m_i} (\mathbf{J}_{vm_i}^0)^T \mathbf{J}_{vm_i}^0 + (\mathbf{J}_{\omega_{m_i}}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{J}_{\omega_{m_i}}^{M_i}) \dot{\mathbf{q}} \\ &\quad + (\mathbf{z}_0)^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i}^2 \dot{\theta}_i^2 \\ &\quad + 2 \dot{\mathbf{q}}^T (\mathbf{J}_{\omega_{m_i}}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i} \dot{\theta}_i) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \dot{\mathbf{q}}^T \left(\sum_{i=1}^N (m_{m_i} (\mathbf{J}_{vm_i}^0)^T \mathbf{J}_{vm_i}^0 + (\mathbf{J}_{\omega m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{J}_{\omega m_i}^{M_i}) \right) \dot{\mathbf{q}} \\
&+ \frac{1}{2} \sum_{i=1}^N (\mathbf{z}_0)^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i}^2 \dot{\theta}_i^2 \\
&+ \dot{\mathbf{q}}^T \sum_{i=1}^N \left((\mathbf{J}_{\omega m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i} \dot{\theta}_i \right) \\
&= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_m(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta} \quad (15)
\end{aligned}$$

In the Equation (15), $\mathbf{M}_m(\mathbf{q})$ is the $N \times N$ inertia matrix of the motors:

$$\mathbf{M}_m(\mathbf{q}) = \sum_{i=1}^N (m_{m_i} (\mathbf{J}_{vm_i}^0)^T \mathbf{J}_{vm_i}^0 + (\mathbf{J}_{\omega m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{J}_{\omega m_i}^{M_i}) \quad (16)$$

The matrix $\mathbf{B} = \text{diag}(B_1, B_2, \dots, B_N)$ is the $N \times N$ constant diagonal inertia matrix of the motors (reflected through the reduction ratios) collecting the motors inertial components around the rotation axes, whose diagonal elements are defined as (noticing that $\mathbf{I}_{m_i}^{M_i}$ is a diagonal matrix, and $\mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 = I_{m_i} \mathbf{z}_0$):

$$B_i = (\mathbf{z}_0)^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i}^2 = \eta_{m_i}^2 I_{m_i} \quad (17)$$

The $N \times N$ matrix $\mathbf{S}(\mathbf{q}) = [\mathbf{S}_1(\mathbf{q}) \ \mathbf{S}_2(\mathbf{q}) \ \dots \ \mathbf{S}_N(\mathbf{q})]$ describes the inertial coupling between the links and the actuators whose columns are defined as:

$$\mathbf{S}_i(\mathbf{q}) = (\mathbf{J}_{\omega m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i} = \eta_{m_i} I_{m_i} (\mathbf{J}_{\omega m_i}^{M_i})^T \mathbf{z}_0 \quad (18)$$

C. LAGRANGIAN EQUATIONS

Now the dynamic equations of robotic manipulators with all elastic joints are derived. From Equations (8) and (15), the kinetic energy is defined as:

$$\begin{aligned}
\mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\theta}) &= \mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\theta}) \\
&= \frac{1}{2} \dot{\mathbf{q}}^T (\mathbf{M}_m(\mathbf{q}) + \mathbf{M}_l(\mathbf{q})) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta} \\
&= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta}
\end{aligned}$$

where

$$\mathbf{M}(\mathbf{q}) = \mathbf{M}_m(\mathbf{q}) + \mathbf{M}_l(\mathbf{q})$$

The Lagrangian of the robotic manipulator is defined as:

$$\begin{aligned}
\mathbb{L}(\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta}) &= \mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\theta}) - \mathbb{U}(\mathbf{q}, \theta) \\
&= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta} - \mathbb{U}_g(\mathbf{q}) - \mathbb{U}_e(\mathbf{q}, \theta) \quad (19)
\end{aligned}$$

From now on, $\mathbb{L}(\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta})$ will be referred to as $\mathbb{L}$ for short. Euler-Lagrange's equations are then expressed as follows:

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathbb{L}}{\partial \mathbf{q}} = \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathbb{L}}{\partial \theta} = \boldsymbol{\tau}_m + \boldsymbol{\tau}_{fr,m} \end{cases} \quad (20)$$

where $\boldsymbol{\tau}_{ex}$, $\boldsymbol{\tau}_{fr,l}$, $\boldsymbol{\tau}_{fr,m}$, and $\boldsymbol{\tau}_m$ are vectors of generalised forces from the external forces/torques, friction torques in the joints,

friction torques in the motors, and motor torques, respectively. Let $\mathbf{f}_e^0$ and $\boldsymbol{\tau}_e^0$ denote the external force and torque vectors applied to the end-effector, expressed in the base frame 0, respectively. Then:

$$\boldsymbol{\tau}_{ex} = (\mathbf{J}_e)^T \mathbf{w}_e \quad (21)$$

where $\mathbf{w}_e^0 = [\mathbf{f}_e^{0T} \ \boldsymbol{\tau}_e^{0T}]^T$ is the end-effector wrench, and $\mathbf{J}_e$ is the Jacobian matrix of the end-effector. Let $\mathbf{P}_e^0$ denote the position vector of the origin of the end-effector's coordinate system (frame $N+1$), expressed in the base frame 0. Then:

$$\mathbf{J}_e = \begin{bmatrix} \mathbf{J}_{ve} \\ \mathbf{J}_{\omega e} \end{bmatrix} \quad (22)$$

where

$$\mathbf{J}_{ve} = \begin{bmatrix} \frac{\partial \mathbf{P}_e^0}{\partial q_1} & \frac{\partial \mathbf{P}_e^0}{\partial q_2} & \dots & \frac{\partial \mathbf{P}_e^0}{\partial q_N} \end{bmatrix}$$

and

$$\mathbf{J}_{\omega e} = [\rho_1 \mathbf{z}_1^0 \ \rho_2 \mathbf{z}_2^0 \ \dots \ \rho_N \mathbf{z}_N^0]$$

In this work, only viscous friction is considered when calculating $\boldsymbol{\tau}_{fr,l}$ and $\boldsymbol{\tau}_{fr,m}$. Let $\mathbf{F}_l^v$ and $\mathbf{F}_m^v$ denote the diagonal matrices of viscous coefficients of joints and motors, respectively. Then:

$$\boldsymbol{\tau}_{fr,l} = -\mathbf{F}_l^v \dot{\mathbf{q}} \quad (23)$$

$$\boldsymbol{\tau}_{fr,m} = -\mathbf{F}_m^v \dot{\theta} \quad (24)$$

Each component on the left-hand side of Euler-Lagrange's equations (20) is calculated as follows:

$$\frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\mathbf{q}}} \right) = \frac{d}{dt} (\mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \mathbf{S}(\mathbf{q}) \dot{\theta}) \quad (25)$$

$$= \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \dot{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{q}} + \mathbf{S}(\mathbf{q}) \ddot{\theta} + \dot{\mathbf{S}}(\mathbf{q}) \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\theta}} \right) = \frac{d}{dt} (\mathbf{B} \dot{\theta} + \mathbf{S}(\mathbf{q})^T \dot{\mathbf{q}}) \quad (26)$$

$$= \mathbf{B} \ddot{\theta} + \mathbf{S}(\mathbf{q})^T \ddot{\mathbf{q}} + \dot{\mathbf{S}}(\mathbf{q})^T \dot{\mathbf{q}}$$

$$\begin{aligned}
\frac{\partial \mathbb{L}}{\partial \mathbf{q}} &= \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_1} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_2} \dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_N} \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_1} \dot{\theta} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_2} \dot{\theta} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_N} \dot{\theta} \end{bmatrix} \\
&\quad - \frac{\partial \mathbb{U}_g(\mathbf{q})}{\partial \mathbf{q}} - \frac{\partial \mathbb{U}_e(\mathbf{q}, \theta)}{\partial \mathbf{q}} \quad (27)
\end{aligned}$$

$$\frac{\partial \mathbb{L}}{\partial \theta} = -\frac{\partial \mathbb{U}_e(\mathbf{q}, \theta)}{\partial \theta} \quad (28)$$

We then define Coriolis and centrifugal matrices $C(\mathbf{q}, \dot{\mathbf{q}})$, $C_1(\mathbf{q}, \dot{\mathbf{q}})$, $C_2(\mathbf{q}, \dot{\mathbf{q}})$, and gravitational vector $\mathbf{G}(\mathbf{q})$ as:

$$C(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q}) - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_1} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_2} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}(\mathbf{q})}{\partial q_N} \end{bmatrix} \quad (29)$$

$$C_1(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{S}}(\mathbf{q}) - \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_1} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_2} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{S}(\mathbf{q})}{\partial q_N} \end{bmatrix}, \quad (30)$$

$$C_2(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{S}}(\mathbf{q})^T, \quad \mathbf{G}(\mathbf{q}) = \frac{\partial \mathbb{U}_g(\mathbf{q})}{\partial \mathbf{q}} \quad (31)$$

Then, the dynamic equations of motion for robotic manipulators with all elastic joints are expressed as:

$$\begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + C_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} \\ C_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{K}\boldsymbol{\eta}_l(\boldsymbol{\eta}_l\mathbf{q} - \boldsymbol{\theta}) \\ \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{\eta}_l\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_m + \boldsymbol{\tau}_{fr,m} \end{bmatrix} \quad (32)$$

The dynamics of the manipulator with all elastic joints can be modelled in an explicit form with the joint variables and the positions of the motors of the elastic joints. The effects of joints' damping, stiffness, gravity, and torque transmission ratio are taken into account. The right side of the equation represents the external excitations, friction, and motor torques. The characteristics of the mathematical model of robotic manipulators with all elastic joints are discussed in more detail in [25].

V. MODELLING ROBOTIC MANIPULATORS WITH MIXED RIGID-ELASTIC JOINTS USING LAGRANGIAN METHOD

Now assume that N_e out of N joints are elastic and the rest $N_r = N - N_e$ joints are rigid. The vector of joint variables is $\mathbf{q} = [q_1 \ q_2 \ \dots \ q_N]^T$, and the $N_e \times 1$ vector of motor angular positions of the elastic joints is $\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \dots \ \theta_{N_e}]^T$. Here, θ_k does not represent the angular position of the motor controlling joint k , but the angular position of the motor of the k^{th} elastic joint. For instance, if a robotic manipulator has six joints, where the last 3 of them are elastic joints, then θ_1 controls q_4 , θ_2 controls q_5 , and θ_3 controls q_6 . For ease of reading, let $\theta_{k,i}$ denote the k^{th} elastic joint corresponding to joint i of the robotic manipulator instead of θ_k where needed. Let $\mathbf{q}_e = [q_{e_1} \ q_{e_2} \ \dots \ q_{e_{N_e}}]^T$ denote the $N_e \times 1$ vector containing only joint variables of elastic joints, which corresponds with

$\boldsymbol{\theta}$ vector. According to the previous example, the vector $\mathbf{q}_e$ is $\mathbf{q}_e = [q_{e_1} \ q_{e_2} \ q_{e_3}]^T = [q_4 \ q_5 \ q_6]^T$.

A. POTENTIAL ENERGY

The gravitational potential energy and the elastic potential energy of robotic manipulators with mixed rigid-elastic joints are similarly calculated as in the case of the robotic manipulator with all elastic joints:

$$\mathbb{U}_g(\mathbf{q}) = - \sum_{i=1}^N (m_{m_i}(\mathbf{g}^0)^T \mathbf{P}_{m_i}^0 + m_{l_i}(\mathbf{g}^0)^T \mathbf{P}_{l_i}^0) \quad (33)$$

$$\mathbb{U}_e(\mathbf{q}_e, \boldsymbol{\theta}) = \frac{1}{2} (\boldsymbol{\eta}_l \mathbf{q}_e - \boldsymbol{\theta})^T \mathbf{K} (\boldsymbol{\eta}_l \mathbf{q}_e - \boldsymbol{\theta}) \quad (34)$$

where $\mathbf{K}$ and $\boldsymbol{\eta}_l$ are $N_e \times N_e$ constant diagonal matrices of joint stiffness and transmission ratio of the torque transmission, respectively.

B. KINETIC ENERGY

Kinetic Energy of the Links

The kinetic energy of the links is solely dependent on the motion of the links. Therefore, it is identical to the case of the robotic manipulator with all elastic joints:

$$\begin{aligned} \mathbb{T}_{link}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} \sum_{i=1}^N (m_{l_i}(\mathbf{v}_{cl_i}^0)^T \mathbf{v}_{cl_i}^0 + (\boldsymbol{\omega}_{l_i}^{i+1})^T \mathbf{I}_{l_i}^{i+1} \boldsymbol{\omega}_{l_i}^{i+1}) \\ &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_l(\mathbf{q}) \dot{\mathbf{q}} \end{aligned} \quad (35)$$

Kinetic Energy of the Motors

The kinetic energy of the motors depends on the motion of the motors:

$$\begin{aligned} \mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) &= \\ \frac{1}{2} \sum_{i=1}^N (m_{m_i}(\mathbf{v}_{cm_i}^0)^T \mathbf{v}_{cm_i}^0 + (\boldsymbol{\omega}_{m_i}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \boldsymbol{\omega}_{m_i}^{M_i}) \end{aligned} \quad (36)$$

The translational velocity vector $\mathbf{v}_{cm_i}^0$ depends on the motion of the centres of mass of the motors. Therefore, they are calculated identically to the case of the robotic manipulator with all elastic joints:

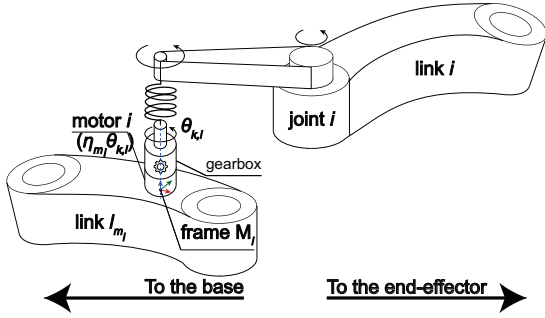
$$\mathbf{v}_{cm_i}^0 = \mathbf{J}_{vm_i}^0 \dot{\mathbf{q}} \quad (37)$$

where

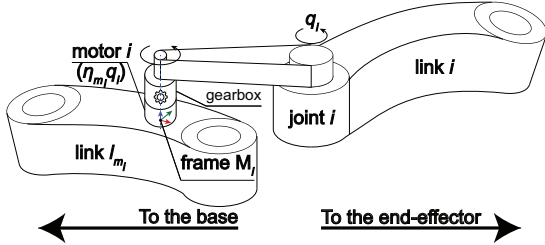
$$\mathbf{J}_{vm_i}^0 = \begin{bmatrix} \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_1} & \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_2} & \dots & \frac{\partial \mathbf{P}_{m_i}^0}{\partial q_{l_{m_i}}} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

The angular velocity vector $\boldsymbol{\omega}_{m_i}^{M_i}$ is calculated through the Jacobian matrix according to the position of the motor and the type of joint it is driving. The angular velocity of a motor is the sum of the angular velocity of the link where the motor is attached and the rotational movement of the motor itself. If joint i is the k^{th} elastic joint, as illustrated in Figure 4a, then the angular velocity of motor i is $\dot{\theta}_{k,i}$. Hence:

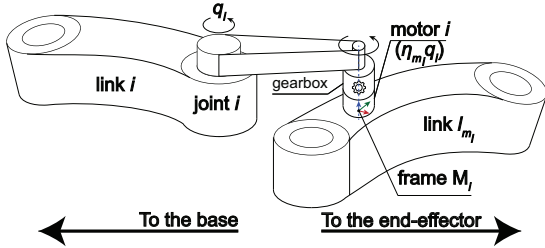
$$\boldsymbol{\omega}_{m_i}^{M_i} = \mathbf{J}_{\omega m_i}^{M_i} \dot{\mathbf{q}} + \mathbf{z}_0 \boldsymbol{\eta}_{m_i} \dot{\theta}_{k,i} \quad (38)$$



(a) Elastic joint i , motor i is placed on link l_{m_i} , which can be any link of the robot.



(b) Rigid joint i , motor i is placed on the link preceding the driven link ($l_{m_i} < i$).



(c) Rigid joint i , motor i is placed on the link after the driven link ($l_{m_i} \geq i$).

FIGURE 4: Motor placement of different joint types on the robotic manipulator.

where

$$\begin{aligned} J_{\omega m_i}^{M_i} &= (R_{M_i}^0)^T J_{\omega m_i}^0 \\ &= (R_{M_i}^0)^T \begin{bmatrix} \rho_1 z_1^0 & \rho_2 z_2^0 & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^0 & 0 & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \rho_1 z_1^{M_i} & \rho_2 z_2^{M_i} & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^{M_i} & 0 & \dots & 0 \end{bmatrix} \end{aligned} \quad (39)$$

If joint i is a rigid joint, then the angular velocity of motor i is $\dot{q}_i$. Hence:

$$\omega_{m_i}^{M_i} = J_{\omega m_i}^{M_i} \dot{q}_i \quad (40)$$

where the Jacobian matrix $J_{\omega m_i}^{M_i}$ is defined according to the position of the motor. If motor i is mounted on the link preceding the driven link ($l_{m_i} < i$), as illustrated in Figure 4b, then:

$$\begin{aligned} J_{\omega m_i}^{M_i} &= (R_{M_i}^0)^T J_{\omega m_i}^0 \\ &= (R_{M_i}^0)^T \begin{bmatrix} \rho_1 z_1^0 & \rho_2 z_2^0 & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^0 & 0 & \dots & z_0^0 \eta_{m_i} & \dots & 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \rho_1 z_1^{M_i} & \rho_2 z_2^{M_i} & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^{M_i} & 0 & \dots & z_0^0 \eta_{m_i} & \dots & 0 \end{bmatrix} \quad (41)$$

Here, the vector $z_0^0 \eta_{m_i}$ is added at the i^{th} column in the $J_{\omega m_i}^{M_i}$ matrix.

If motor i is mounted on the link after the driven link ($l_{m_i} \geq i$), as illustrated in Figure 4c, then:

$$\begin{aligned} J_{\omega m_i}^{M_i} &= (R_{M_i}^0)^T J_{\omega m_i}^0 \\ &= (R_{M_i}^0)^T \begin{bmatrix} \rho_1 z_1^0 & \rho_2 z_2^0 & \dots & \left(\rho_i z_i^0 + z_0^0 \eta_{m_i} \right) & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^0 & 0 & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \rho_1 z_1^{M_i} & \rho_2 z_2^{M_i} & \dots & \left(\rho_i z_i^{M_i} + z_0^0 \eta_{m_i} \right) & \dots & \rho_{l_{m_i}} z_{l_{m_i}}^{M_i} & 0 & \dots & 0 \end{bmatrix} \end{aligned} \quad (42)$$

Examples will be provided to illustrate how the Jacobian matrices are computed for the three cases in Equations (39), (41), and (42). Consider a three-DOF robotic manipulator. If joint 2 is an elastic joint, and its motor is attached to link 1, then the Jacobian matrix of motor 2 is:

$$J_{\omega m_2}^{M_2} = [\rho_1 z_1^{M_2} \quad 0 \quad 0] \quad (43)$$

If joint 2 is a rigid joint, and its motor is attached to link 1, then the Jacobian matrix of motor 2 is:

$$J_{\omega m_2}^{M_2} = [\rho_1 z_1^{M_2} \quad z_0^0 \eta_{m_2} \quad 0] \quad (44)$$

If joint 2 is a rigid joint, and its motor is attached to link 2, then the Jacobian matrix of motor 2 is:

$$J_{\omega m_2}^{M_2} = [\rho_1 z_1^{M_2} \quad (\rho_2 z_2^{M_2} + z_0^0 \eta_{m_2}) \quad 0] \quad (45)$$

If joint 2 is a rigid joint, and its motor is attached to link 3, then the Jacobian matrix of motor 2 is:

$$J_{\omega m_2}^{M_2} = [\rho_1 z_1^{M_2} \quad (\rho_2 z_2^{M_2} + z_0^0 \eta_{m_2}) \quad \rho_3 z_3^{M_2}] \quad (46)$$

Let $\sum_r$ and $\sum_e$ denote the summations including all rigid joints and all elastic joints, respectively. The kinetic energy of motors becomes:

$$\begin{aligned} \mathbb{T}_{motor}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) &= \frac{1}{2} \sum_{i=1}^N m_{m_i} \dot{\mathbf{q}}^T (J_{vm_i}^0)^T J_{vm_i}^0 \dot{\mathbf{q}} \\ &+ \frac{1}{2} \sum_r \dot{\mathbf{q}}^T (J_{\omega m_i}^{M_i})^T I_{m_i}^{M_i} J_{\omega m_i}^{M_i} \dot{\mathbf{q}} \\ &+ \frac{1}{2} \sum_e \left[\left(\dot{\mathbf{q}}^T (J_{\omega m_i}^{M_i})^T + (z_0^0)^T \eta_{m_i} \dot{\theta}_{k,i} \right) I_{m_i}^{M_i} \left(J_{\omega m_i}^{M_i} \dot{\mathbf{q}} + z_0^0 \eta_{m_i} \dot{\theta}_{k,i} \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^N m_{m_i} \dot{\mathbf{q}}^T (J_{vm_i}^0)^T J_{vm_i}^0 \dot{\mathbf{q}} \\ &+ \frac{1}{2} \sum_r \dot{\mathbf{q}}^T (J_{\omega m_i}^{M_i})^T I_{m_i}^{M_i} J_{\omega m_i}^{M_i} \dot{\mathbf{q}} \\ &+ \frac{1}{2} \sum_e \dot{\mathbf{q}}^T (J_{\omega m_i}^{M_i})^T I_{m_i}^{M_i} J_{\omega m_i}^{M_i} \dot{\mathbf{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_e \left((\mathbf{z}_0)^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i}^2 \dot{\theta}_{k,i}^2 + 2 \dot{\mathbf{q}}^T (\mathbf{J}_{\omega_{m_i}}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{z}_0 \eta_{m_i} \dot{\theta}_{k,i} \right) \\
& = \frac{1}{2} \dot{\mathbf{q}}^T \left[\sum_{i=1}^N (m_{m_i} (\mathbf{J}_{v_{m_i}}^0)^T \mathbf{J}_{v_{m_i}}^0 + (\mathbf{J}_{\omega_{m_i}}^{M_i})^T \mathbf{I}_{m_i}^{M_i} \mathbf{J}_{\omega_{m_i}}^{M_i}) \right] \dot{\mathbf{q}} \\
& + \frac{1}{2} \sum_e I_{m_{z_i}} \eta_{m_i}^2 \dot{\theta}_{k,i}^2 + \dot{\mathbf{q}}^T \sum_e \left((\mathbf{J}_{\omega_{m_i}}^{M_i})^T \mathbf{z}_0 I_{m_{z_i}} \eta_{m_i} \dot{\theta}_{k,i} \right) \\
& = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_m(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta} \quad (47)
\end{aligned}$$

where the matrix $\mathbf{M}_m(\mathbf{q})$ has an $N \times N$ size, the inertia matrix $\mathbf{B}$ has an $N_e \times N_e$ size, and the coupling matrix $\mathbf{S}(\mathbf{q})$ has an $N \times N_e$ size, and they are similarly calculated as in the case of the robotic manipulator with all elastic joints. However, notice that now, only elastic joints contribute to matrices $\mathbf{B}$ and $\mathbf{S}(\mathbf{q})$.

C. LAGRANGIAN EQUATIONS

Now the dynamic equations of a robotic manipulator with mixed rigid-elastic joints are derived. Euler-Lagrange's equations are written as follows:

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathbb{L}}{\partial \mathbf{q}} = \boldsymbol{\tau}_{mr} + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathbb{L}}{\partial \theta} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{cases} \quad (48)$$

where $\boldsymbol{\tau}_{mr} = [\tau_{mr1} \ \tau_{mr2} \ \dots \ \tau_{mrN}]^T$ is the $N \times 1$ vector of motor torques of rigid joints, and $\boldsymbol{\tau}_{me}$ is the $N_e \times 1$ vector of motor torques of elastic joints, and

$$\tau_{mr_i} = \begin{cases} 0 & \text{if the joint } i \text{ is an elastic joint} \\ \tau_{m_i}(t) & \text{if the joint } i \text{ is a rigid joint} \end{cases}$$

$\boldsymbol{\tau}_{ex}$ is the $N \times 1$ vector of external torques acting on the joints and is calculated similarly to the case of the robotic manipulator with all elastic joints. $\boldsymbol{\tau}_{fr,l}$ and $\boldsymbol{\tau}_{fr,m}$ are $N \times 1$ vector of viscous friction torques in the joints and $N_e \times 1$ vector of viscous friction torques in the motors of elastic joints, respectively. Notice that friction torques in the motors of rigid joints are included in the $\boldsymbol{\tau}_{fr,l}$ vector. In particular, if joint i is a rigid joint, then:

$$\tau_{fr,l_i} = (-F_{l_i}^v - F_{m_i}^v) \dot{q}_i$$

and if joint i is the k^{th} elastic joint, then:

$$\begin{aligned}
\tau_{fr,l_i} &= -F_{l_i}^v \dot{q}_i \\
\tau_{fr,m_k} &= -F_{m_i}^v \dot{\theta}_{k,i}
\end{aligned}$$

where τ_{fr,l_i} and τ_{fr,m_k} are the i^{th} components of the $\boldsymbol{\tau}_{fr,l}$ and $\boldsymbol{\tau}_{fr,m}$ vectors, and $F_{l_i}^v$ and $F_{m_i}^v$ are friction coefficients.

Then, doing the same calculation steps as in the case of the robotic manipulator with all elastic joints, the Euler-Lagrange's equations become:

$$\begin{aligned}
& \begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}}) \dot{\theta} \\ \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix} \\
& + \begin{bmatrix} -\boldsymbol{\tau}_{elas}^q \\ \boldsymbol{\tau}_{elas} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{mr} + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{bmatrix} \quad (49)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\tau}_{elas}^q &= \frac{\partial \mathbb{U}_e(\mathbf{q}_e, \boldsymbol{\theta})}{\partial \mathbf{q}} = [\tau_{elas1}^q \ \tau_{elas2}^q \ \dots \ \tau_{elasN}^q]^T, \\
\tau_{elas_i}^q &= \begin{cases} K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \text{if joint } i \text{ is the } k^{th} \text{ elastic joint} \\ 0 & \text{if joint } i \text{ is a rigid joint} \end{cases} \\
\boldsymbol{\tau}_{elas} &= \frac{\partial \mathbb{U}_e(\mathbf{q}_e, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{\eta} \mathbf{q}_e)
\end{aligned}$$

and $K_{k,i}$ is the stiffness of the k^{th} elastic joint, which corresponding to the i^{th} joint of the robotic manipulator.

Note that $\boldsymbol{\tau}_{elas}^q$ and $\boldsymbol{\tau}_{elas}$ represent the elastic forces affecting both the load and motor sides of elastic joints. In addition, the load angular positions and load forces of elastic joints can be controlled by adjusting the deviation between the motor and load angular positions (i.e. elastic forces $\boldsymbol{\tau}_{elas}^q$). On the other hand, the load angular positions and load force of rigid joints are adjusted by directly controlling the torques of the motor (i.e. $\boldsymbol{\tau}_{mr}$). Therefore, it is convenient to gather the elastic forces from elastic joints and the motor torques from the rigid joints into a vector. Let $\boldsymbol{\tau}_q$ denote the vector of load side torques, and:

$$\begin{aligned}
\boldsymbol{\tau}_q &= \boldsymbol{\tau}_{mr} + \boldsymbol{\tau}_{elas}^q = [\tau_{q1} \ \dots \ \tau_{qN}]^T \\
\tau_{q_i} &= \begin{cases} K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \text{if joint } i \text{ is the } k^{th} \text{ elastic joint} \\ \tau_{m_i}(t) & \text{if joint } i \text{ is a rigid joint} \end{cases} \quad (50)
\end{aligned}$$

Then, the Euler-Lagrange's equations become:

$$\begin{aligned}
& \begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}}) \dot{\theta} \\ \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix} \\
& + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\tau}_{elas} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_q + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{bmatrix} \quad (51)
\end{aligned}$$

There are numerous similarities as well as differences in the dynamic equations of robotic manipulators with all elastic joints (Equations (32)) and of robotic manipulators with mixed rigid-elastic joints (Equations (51)). First of all, the vector of joint variables $\mathbf{q}$ is an $N \times 1$ vector in both cases. However, the vector of motor angular positions $\boldsymbol{\theta}$ is an $N \times 1$ in the Equations (32), while it is an $N_e \times 1$ vector in the Equations (51). The mass matrix $\mathbf{M}(\mathbf{q}) = \mathbf{M}_m(\mathbf{q}) + \mathbf{M}_l(\mathbf{q})$ is a square $N \times N$ matrix in both Equations (32) and Equations (51). The matrix $\mathbf{M}_l(\mathbf{q})$ is calculated identically in both cases, as shown in Equations (9) and (35). Although the calculation of the matrix $\mathbf{M}_m(\mathbf{q})$ seems to be similar (Equations (16) and (47) for robotic manipulators with all elastic joints and with mixed rigid-elastic joints, respectively), some differences are coming from the way the Jacobian matrices $\mathbf{J}_{\omega_{m_i}}^{M_i}$ are calculated. In particular, in the case where all joints are elastic, Jacobian matrices $\mathbf{J}_{\omega_{m_i}}^{M_i}$ are calculated by Equation (14), while in the case of mixed rigid-elastic joints, Jacobian matrices

$J_{\omega m_i}^{M_i}$ are calculated by Equations (39), (41), or (42) depending on the joint type and the position of the motor.

The calculation of matrices $S(q)$ and B only uses the parameters of the elastic joints. Therefore, in the case of robotic manipulators with all elastic joints, these matrices are square $N \times N$ matrices, where their elements are calculated from Equations (17) and (18). However, in the case of robotic manipulators with mixed rigid-elastic joints, matrix $S(q)$ is an $N \times N_e$ matrix, and matrix B is a square $N_e \times N_e$ matrix, where their elements are calculated similarly in the case of all elastic joints.

The Coriolis and centrifugal matrices $C(q, \dot{q})$, $C_1(q, \dot{q})$, $C_2(q, \dot{q})$, and gravitational vector $G(q)$ are calculated identically from Equations (29), (30), (31) for both types of robotic manipulators. These matrices are square $N \times N$ matrices in the case of robotic manipulators with all elastic joints, while in the case of robotic manipulators with mixed rigid-elastic joints, $C(q, \dot{q})$ is an $N \times N$ matrix, $C_1(q, \dot{q})$ is an $N \times N_e$ matrix, and $C_2(q, \dot{q})$ is an $N_e \times N$ matrix.

The elastic torques are calculated quite similarly in both Equations (32) and (51), which are $K(\theta - \eta/q)$ in Equations (32) and $\tau_{elas} = K(\theta - \eta/q_e)$ in Equations (51). They are gathered in the lower part of the fourth term in both equations. This term represents the elastic torques directly affecting the motors of elastic joints. The reaction torques, which affect the output of the elastic joints, are gathered in the upper part of the fourth term in Equations (32) and in vector τ_q in Equations (51) as shown in Equation (50). In robotic manipulators with all elastic joints, the motor torques, collected in vector τ_m in Equations (32), drive the elastic joints indirectly through the elastic torques. In robotic manipulators with mixed rigid-elastic joints, the motor torques of rigid joints, collected in vector τ_q in Equations (51), directly drive the rigid joints, while the motor torques of elastic joints, collected in vector τ_{me} in Equations (51), still drive the elastic joints indirectly through the elastic torques.

VI. SKEW-SYMMETRIC PROPERTY

One of the useful properties is the skew-symmetry of the matrix $W = \dot{M} - 2C$. If the element C_{ij} of matrix C is defined using the Christoffel symbols:

$$C_{ij} = \sum_{k=1}^N C_{ijk} \dot{q}_k \quad (52)$$

$$C_{ijk} = \frac{1}{2} \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \quad (53)$$

then matrix W is skew-symmetric [34].

Let $\mathcal{M}$ and $\mathcal{C}$ denote the mass matrix and the Coriolis and centrifugal matrix of the whole dynamic equations of robotic manipulators with mixed rigid-elastic joints (51), where:

$$\mathcal{M}(q) = \begin{bmatrix} M(q) & S(q) \\ S(q)^T & B \end{bmatrix}$$

$$\mathcal{C}(q, \dot{q}) = \begin{bmatrix} C(q, \dot{q}) & C_1(q, \dot{q}) \\ C_2(q, \dot{q}) & 0 \end{bmatrix}$$

Let $\Theta = [q^T \theta^T]^T$ denote the vector collect all the joint and motor variables. From the definitions of C , C_1 , and C_2 from (29), (30), and (31), one has:

$$\begin{aligned} \mathcal{C}\dot{\Theta} &= \begin{bmatrix} C\dot{q} + C_1\dot{\theta} \\ C_2\dot{q} \end{bmatrix} \\ &= \begin{bmatrix} \dot{M}\dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_N} \end{bmatrix} \dot{q} + \dot{S}\dot{\theta} - \begin{bmatrix} \dot{q}^T \frac{\partial S}{\partial q_1} \\ \vdots \\ \dot{q}^T \frac{\partial S}{\partial q_N} \end{bmatrix} \dot{\theta} \\ \dot{S}^T \dot{q} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\mathcal{M}} - \frac{1}{2} \begin{bmatrix} \dot{\Theta}^T \frac{\partial \mathcal{M}}{\partial \Theta_1} \\ \vdots \\ \dot{\Theta}^T \frac{\partial \mathcal{M}}{\partial \Theta_{N+N_e}} \end{bmatrix} \dot{\Theta} \end{bmatrix} \dot{\Theta} \quad (54) \end{aligned}$$

The $\mathcal{M}$ and $\mathcal{C}$ matrices have the same relationship as the M and C matrices. In addition, the matrix $\mathcal{M}$ is symmetric. From the equation above, the i^{th} element of the vector $\mathcal{C}\dot{\Theta}$ can be expressed as:

$$\begin{aligned} \sum_{j=1}^{N+N_e} C_{ij} \dot{\Theta}_j &= \sum_{j=1}^{N+N_e} \left(\dot{M}_{ij} - \frac{1}{2} \sum_{k=1}^{N+N_e} \frac{\partial M_{kj}}{\partial \Theta_i} \dot{\Theta}_k \right) \dot{\Theta}_j \\ &= \sum_{j=1}^{N+N_e} \sum_{k=1}^{N+N_e} \left(\frac{\partial M_{ij}}{\partial \Theta_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \Theta_i} \right) \dot{\Theta}_k \dot{\Theta}_j \\ &= \frac{1}{2} \sum_{j=1}^{N+N_e} \sum_{k=1}^{N+N_e} \left(\frac{\partial M_{ij}}{\partial \Theta_k} + \frac{\partial M_{ik}}{\partial \Theta_j} - \frac{\partial M_{jk}}{\partial \Theta_i} \right) \dot{\Theta}_k \dot{\Theta}_j \quad (55) \end{aligned}$$

If the element C_{ij} of matrix $\mathcal{C}$ is calculated as:

$$C_{ij} = \frac{1}{2} \sum_{k=1}^{N+N_e} \left(\frac{\partial M_{ij}}{\partial \Theta_k} + \frac{\partial M_{ik}}{\partial \Theta_j} - \frac{\partial M_{jk}}{\partial \Theta_i} \right) \dot{\Theta}_k \quad (56)$$

then, the matrix $\mathcal{W} = \dot{\mathcal{M}} - 2\mathcal{C}$ is skew-symmetric. The skew-symmetric property of the $\dot{\mathcal{M}} - 2\mathcal{C}$ matrix implies several important meanings, such as the energy conservation of the Coriolis and centrifugal forces, enabling the derivation of passivity-based controllers, as well as simplifying Lyapunov proofs.

VII. LINEARITY OF THE DYNAMIC MATHEMATICAL MODEL WITH RESPECT TO THE DYNAMIC PARAMETERS

Similar to the dynamic model of a robotic manipulator with all rigid joints, the dynamic model of a robotic manipulator with mixed rigid-elastic joints is also linear with respect to vectors of dynamic parameters, i.e. the dynamic formulation can be expressed as a function of vectors of dynamic parameters and regression matrices. In this section, we derive formulas to calculate these vectors and matrices.

Link i and all motors attached to link i are unified into a rigid body (unified rigid body i), and let $\mathbf{P}_{c_i}^0$ denote the position vector of the centre of mass of that unified rigid body, expressed in the base frame, as shown in the Figure 5. Let $\mathbf{r}_{cl_i}^0$ and $\mathbf{r}_{cm_j}^0$ denote the vectors pointing from $\mathbf{P}_{c_i}^0$ to the centre of mass of link i ($\mathbf{P}_{l_i}^0$) and to the centre of mass of motor j attached to link i ($\mathbf{P}_{m_j}^0$), respectively. Notice that, when expressed in frame $i+1$, vectors $\mathbf{r}_{cl_i}^{i+1}$ and $\mathbf{r}_{cm_j}^{i+1}$ are constants because the frame $i+1$ is rigidly attached to link i . Then, the position of the centre of mass of the link i and motor j can be expressed with reference to the centre of mass of the unified rigid body as:

$$\mathbf{P}_{l_i}^0 = \mathbf{P}_{c_i}^0 + \mathbf{r}_{cl_i}^0 = \mathbf{P}_{c_i}^0 + \mathbf{R}_{i+1}^0 \mathbf{r}_{cl_i}^{i+1} \quad (57)$$

$$\mathbf{P}_{m_j}^0 = \mathbf{P}_{c_i}^0 + \mathbf{r}_{cm_j}^0 = \mathbf{P}_{c_i}^0 + \mathbf{R}_{i+1}^0 \mathbf{r}_{cm_j}^{i+1} \quad (58)$$

Taking the time derivative of $\mathbf{P}_{l_i}^0$ and $\mathbf{P}_{m_j}^0$ gives

$$\dot{\mathbf{P}}_{l_i}^0 = \dot{\mathbf{P}}_{c_i}^0 + \dot{\mathbf{R}}_{i+1}^0 \mathbf{r}_{cl_i}^{i+1} = \dot{\mathbf{P}}_{c_i}^0 + [\boldsymbol{\omega}_i^0]_{\times} \mathbf{r}_{cl_i}^0 \quad (59)$$

$$\dot{\mathbf{P}}_{m_j}^0 = \dot{\mathbf{P}}_{c_i}^0 + \dot{\mathbf{R}}_{i+1}^0 \mathbf{r}_{cm_j}^{i+1} = \dot{\mathbf{P}}_{c_i}^0 + [\boldsymbol{\omega}_i^0]_{\times} \mathbf{r}_{cm_j}^0 \quad (60)$$

where $[\mathbf{a}]_{\times}$ denote the skew-symmetric matrix of vector $\mathbf{a}$. Furthermore, $\boldsymbol{\omega}_i^0 = \boldsymbol{\omega}_{l_i}^0$ is the angular velocity of frame $i+1$ and also the angular velocity of the unified rigid body i because frame $i+1$ rigidly attached link i .

Substituting Equation (59) into the kinetic energy expression of link i in the Equation (35) gives:

$$\begin{aligned} \mathbb{T}_{link,i}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} \left(m_{l_i} (\dot{\mathbf{P}}_{l_i}^0)^T \dot{\mathbf{P}}_{l_i}^0 + (\boldsymbol{\omega}_i^0)^T \mathbf{I}_{l_i}^0 \boldsymbol{\omega}_i^0 \right) \\ &= \frac{1}{2} m_{l_i} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} m_{l_i} \mathbf{r}_{cl_i}^0 \\ &\quad + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T m_{l_i} [\mathbf{r}_{cl_i}^0]_{\times} [\mathbf{r}_{cl_i}^0]_{\times} \boldsymbol{\omega}_i^0 + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T \mathbf{I}_{l_i}^0 \boldsymbol{\omega}_i^0 \\ &= \frac{1}{2} m_{l_i} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} m_{l_i} \mathbf{r}_{cl_i}^0 + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T \bar{\mathbf{I}}_{l_i}^0 \boldsymbol{\omega}_i^0 \end{aligned} \quad (61)$$

where

$$\bar{\mathbf{I}}_{l_i}^0 = m_{l_i} [\mathbf{r}_{cl_i}^0]_{\times}^T [\mathbf{r}_{cl_i}^0]_{\times} + \mathbf{I}_{l_i}^0$$

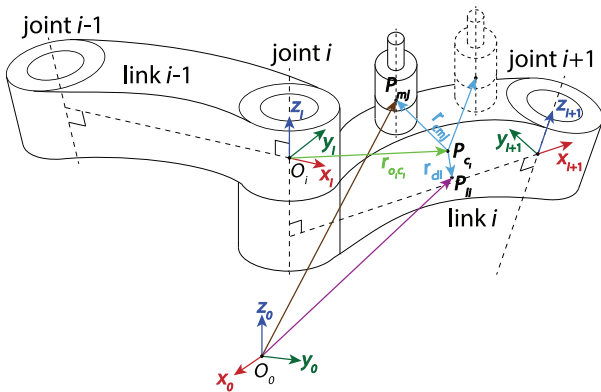


FIGURE 5: Unified rigid body i including link i and all motors attached to it, with the centre of mass $\mathbf{P}_{c_i}^0$.

represents the inertia tensor relative to the overall centre of mass $\mathbf{P}_{c_i}^0$. Notice that when there is no motor attached to link i , $\mathbf{P}_{c_i}^0 \equiv \mathbf{P}_{l_i}^0$, $\mathbf{r}_{cl_i}^0 = \mathbf{0}$, and $\bar{\mathbf{I}}_{l_i}^0 = \mathbf{I}_{l_i}^0$.

Now, consider motor j attached to link i , if joint j driven by motor j is the k^{th} elastic joint, then the angular velocity of the motor is $\dot{\theta}_{k,j}$. Thus:

$$\boldsymbol{\omega}_{m_j}^0 = \boldsymbol{\omega}_i^0 + \mathbf{z}_{m_j}^0 \eta_{m_j} \dot{\theta}_{k,j} \quad (62)$$

and if joint j driven by motor j is a rigid joint, then the angular velocity of the motor is $\dot{q}_j$. Thus:

$$\boldsymbol{\omega}_{m_j}^0 = \boldsymbol{\omega}_i^0 + \mathbf{z}_{m_j}^0 \eta_{m_j} \dot{q}_j \quad (63)$$

By substituting Equations (60), (62), (63) into the kinetic energy expression of motors (36), and denoting $\sum_{m \in \text{link } i}$ as the sum including all motors attached to link i , $\sum_{mr \in \text{link } i}$ as the sum including all motors attached to link i which are driving rigid joints, and $\sum_{me \in \text{link } i}$ as the sum including all motors attached to link i which are driving elastic joints, the kinetic energy of all motors attached to link i is:

$$\begin{aligned} \mathbb{T}_{motor,i} &= \frac{1}{2} \sum_{m \in \text{link } i} \left(m_{m_j} (\dot{\mathbf{P}}_{m_j}^0)^T \dot{\mathbf{P}}_{m_j}^0 + (\boldsymbol{\omega}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_{m_j}^0 \right) \\ &= \sum_{m \in \text{link } i} \left(\frac{1}{2} m_{m_j} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} m_{m_j} \mathbf{r}_{cm_j}^0 \right. \\ &\quad \left. + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T m_{m_j} ([\mathbf{r}_{cm_j}^0]_{\times})^T [\mathbf{r}_{cm_j}^0]_{\times} \boldsymbol{\omega}_i^0 \right) \\ &\quad + \sum_{mr \in \text{link } i} \frac{1}{2} (\boldsymbol{\omega}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_{m_j}^0 + \sum_{me \in \text{link } i} \frac{1}{2} (\boldsymbol{\omega}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_{m_j}^0 \\ &= \sum_{m \in \text{link } i} \left(\frac{1}{2} m_{m_j} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} m_{m_j} \mathbf{r}_{cm_j}^0 \right. \\ &\quad \left. + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T m_{m_j} ([\mathbf{r}_{cm_j}^0]_{\times})^T [\mathbf{r}_{cm_j}^0]_{\times} \boldsymbol{\omega}_i^0 \right) \\ &\quad + \sum_{mr \in \text{link } i} \frac{1}{2} \left((\boldsymbol{\omega}_i^0)^T + (\mathbf{z}_{m_j}^0)^T \eta_{m_j} \dot{q}_j \right) \mathbf{I}_{m_j}^0 \left(\boldsymbol{\omega}_i^0 + \mathbf{z}_{m_j}^0 \eta_{m_j} \dot{q}_j \right) \\ &\quad + \sum_{me \in \text{link } i} \frac{1}{2} \left((\boldsymbol{\omega}_i^0)^T + (\mathbf{z}_{m_j}^0)^T \eta_{m_j} \dot{\theta}_{k,j} \right) \mathbf{I}_{m_j}^0 \left(\boldsymbol{\omega}_i^0 + \mathbf{z}_{m_j}^0 \eta_{m_j} \dot{\theta}_{k,j} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{m \in \text{link } i} \left(\frac{1}{2} m_{m_j} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} m_{m_j} \mathbf{r}_{cm_j}^0 \right. \\
&\quad \left. + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T m_{m_j} ([\mathbf{r}_{cm_j}^0]_{\times})^T [\mathbf{r}_{cm_j}^0]_{\times} \boldsymbol{\omega}_i^0 \right) \\
&\quad + \sum_{m \in \text{link } i} \frac{1}{2} (\boldsymbol{\omega}_i^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \\
&\quad + \sum_{mr \in \text{link } i} \left(\eta_{m_j} \dot{q}_j (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{q}_j^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \\
&\quad + \sum_{me \in \text{link } i} \left(\eta_{m_j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{\theta}_{k,j}^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \\
&= \frac{1}{2} \sum_{m \in \text{link } i} \left(m_{m_j} (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 \right) \\
&\quad + (\dot{\mathbf{P}}_{c_i}^0)^T [\boldsymbol{\omega}_i^0]_{\times} \left(\sum_{m \in \text{link } i} m_{m_j} \mathbf{r}_{cm_j}^0 \right) + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T \bar{\mathbf{I}}_{m_i}^0 \boldsymbol{\omega}_i^0 \\
&\quad + \sum_{mr \in \text{link } i} \left(\eta_{m_j} \dot{q}_j (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{q}_j^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \\
&\quad + \sum_{me \in \text{link } i} \left(\eta_{m_j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{\theta}_{k,j}^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \quad (64)
\end{aligned}$$

where

$$\bar{\mathbf{I}}_{m_i}^0 = \sum_{m \in \text{link } i} \left(m_{m_j} ([\mathbf{r}_{cm_j}^0]_{\times})^T [\mathbf{r}_{cm_j}^0]_{\times} + \mathbf{I}_{m_j}^0 \right)$$

Let m_i and $\bar{\mathbf{I}}_i^0$ denote the total mass and the inertia tensor of the unified rigid body i , respectively, and

$$m_i = m_{l_i} + \sum_{m \in \text{link } i} m_{m_j} \quad (65)$$

$$\bar{\mathbf{I}}_i^0 = \bar{\mathbf{I}}_{l_i}^0 + \bar{\mathbf{I}}_{m_i}^0 \quad (66)$$

Furthermore, the position of the centre of mass of the unified rigid body i can be calculated as below:

$$\begin{aligned}
m_i \mathbf{P}_{c_i}^0 &= m_{l_i} \mathbf{P}_{l_i}^0 + \sum_{m \in \text{link } i} m_{m_j} \mathbf{P}_{m_j}^0 \\
&\Leftrightarrow (m_{l_i} + \sum_{m \in \text{link } i} m_{m_j}) \mathbf{P}_{c_i}^0 = m_{l_i} \mathbf{P}_{l_i}^0 + \sum_{m \in \text{link } i} m_{m_j} \mathbf{P}_{m_j}^0 \\
&\Leftrightarrow m_{l_i} (\mathbf{P}_{l_i}^0 - \mathbf{P}_{c_i}^0) + \sum_{m \in \text{link } i} m_{m_j} (\mathbf{P}_{m_j}^0 - \mathbf{P}_{c_i}^0) = \mathbf{0} \\
&\Leftrightarrow m_{l_i} \mathbf{r}_{cl_i}^0 + \sum_{m \in \text{link } i} m_{m_j} \mathbf{r}_{cm_j}^0 = \mathbf{0} \quad (67)
\end{aligned}$$

Then, by summing Equations (61) and (64), and adopting Equations (65), (66), and (67), the kinetic energy of the unified rigid body i is:

$$\begin{aligned}
\mathbb{T}_i &= \mathbb{T}_{\text{link},i} + \mathbb{T}_{\text{motor},i} \\
&= \frac{1}{2} m_i (\dot{\mathbf{P}}_{c_i}^0)^T \dot{\mathbf{P}}_{c_i}^0 + \frac{1}{2} (\boldsymbol{\omega}_i^0)^T \bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0 \\
&\quad + \sum_{mr \in \text{link } i} \left(\eta_{m_j} \dot{q}_j (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{q}_j^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \\
&\quad + \sum_{me \in \text{link } i} \left(\eta_{m_j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \boldsymbol{\omega}_i^0 \right. \\
&\quad \left. + \frac{1}{2} \eta_{m_j}^2 \dot{\theta}_{k,j}^2 (\mathbf{z}_{m_j}^0)^T \mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 \right) \quad (68)
\end{aligned}$$

To retrieve the parameters of the robotic manipulator, the inertia tensor of the unified rigid body i ($\bar{\mathbf{I}}_i^0$) will be represented in frame $i+1$ as $\bar{\mathbf{I}}_i^{i+1}$, and the inertia tensor of motor j attached to the link i ($\mathbf{I}_{m_j}^0$) will be represented in frame M_j as $\mathbf{I}_{m_j}^{M_j}$. Then, $\bar{\mathbf{I}}_i^{i+1}$ and $\mathbf{I}_{m_j}^{M_j}$ contain only constant parameters. Furthermore, as mentioned above, matrix $\mathbf{I}_{m_j}^{M_j}$ is also diagonal. In addition, $\mathbf{z}_{m_j}^0 = \mathbf{z}_0 = [0 \ 0 \ 1]^T$. Then:

$$\begin{aligned}
\mathbf{I}_{m_j}^0 \mathbf{z}_{m_j}^0 &= \mathbf{R}_{M_j}^0 \mathbf{I}_{m_j}^{M_j} (\mathbf{R}_{M_j}^0)^T \mathbf{z}_{m_j}^0 = \mathbf{R}_{M_j}^0 \mathbf{I}_{m_j}^{M_j} \mathbf{z}_{m_j}^{M_j} \\
&= \mathbf{R}_{M_j}^0 \mathbf{I}_{m_{z_j}} \mathbf{z}_{m_j}^{M_j} = \mathbf{I}_{m_{z_j}} \mathbf{z}_{m_j}^0 \quad (69)
\end{aligned}$$

Substituting Equation (69) into Equation (68) and expressing vectors and matrices in frame $i+1$ gives:

$$\begin{aligned}
\mathbb{T}_i &= \frac{1}{2} m_i (\dot{\mathbf{P}}_{c_i}^{i+1})^T \dot{\mathbf{P}}_{c_i}^{i+1} + \frac{1}{2} (\boldsymbol{\omega}_i^{i+1})^T \bar{\mathbf{I}}_i^{i+1} \boldsymbol{\omega}_i^{i+1} \\
&\quad + \sum_{mr \in \text{link } i} \left(\eta_{m_j} \dot{q}_j (\mathbf{z}_{m_j}^{i+1})^T \boldsymbol{\omega}_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 \dot{q}_j^2 \right) \\
&\quad + \sum_{me \in \text{link } i} \left(\eta_{m_j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^{i+1})^T \boldsymbol{\omega}_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 \dot{\theta}_{k,j}^2 \right) \quad (70)
\end{aligned}$$

Let $\mathbf{P}_{o_i}^0$ and $\mathbf{r}_{o_i c_i}^0$ denote the position of the origin of frame i and the vector pointing from $\mathbf{P}_{o_i}^0$ to $\mathbf{P}_{c_i}^0$ expressed in the base frame, respectively. Then:

$$\mathbf{P}_{c_i}^0 = \mathbf{P}_{o_i}^0 + \mathbf{r}_{o_i c_i}^0 = \mathbf{P}_{o_i}^0 + \mathbf{R}_{i+1}^0 \mathbf{r}_{o_i c_i}^{i+1} \quad (71)$$

Note that $\mathbf{r}_{o_i c_i}^{i+1}$ is a constant vector. Then, taking the time derivative of $\mathbf{P}_{c_i}^0$ gives:

$$\dot{\mathbf{P}}_{c_i}^0 = \dot{\mathbf{P}}_{o_i}^0 + \dot{\mathbf{R}}_{i+1}^0 \mathbf{r}_{o_i c_i}^{i+1} = \dot{\mathbf{P}}_{o_i}^0 + [\boldsymbol{\omega}_i^0]_{\times} \mathbf{r}_{o_i c_i}^0 \quad (72)$$

Expressing $\dot{\mathbf{P}}_{c_i}^0$ in frame $i+1$ yields:

$$\dot{\mathbf{P}}_{c_i}^{i+1} = \dot{\mathbf{P}}_{o_i}^{i+1} + [\boldsymbol{\omega}_i^{i+1}]_{\times} \mathbf{r}_{o_i c_i}^{i+1} \quad (73)$$

Substituting Equation (73) into Equation (70) gives:

$$\begin{aligned}
\mathbb{T}_i &= \frac{1}{2} m_i (\dot{\mathbf{P}}_{o_i}^{i+1})^T \dot{\mathbf{P}}_{o_i}^{i+1} \\
&\quad + (\dot{\mathbf{P}}_{o_i}^{i+1})^T [\boldsymbol{\omega}_i^{i+1}]_{\times} m_i \mathbf{r}_{o_i c_i}^{i+1} + \frac{1}{2} (\boldsymbol{\omega}_i^{i+1})^T \bar{\mathbf{I}}_i^{i+1} \boldsymbol{\omega}_i^{i+1}
\end{aligned}$$

$$+ \sum_{mr \in \text{link } i} \left(\eta_{m_j} I_{m_{z_j}} \dot{q}_j (\mathbf{z}_{m_j}^{i+1})^T \boldsymbol{\omega}_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_{z_j}} \dot{q}_j^2 \right) \\ + \sum_{me \in \text{link } i} \left(\eta_{m_j} I_{m_{z_j}} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^{i+1})^T \boldsymbol{\omega}_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_{z_j}} \dot{\theta}_{k,j}^2 \right) \quad (74)$$

where

$$\bar{I}_{o_i}^{i+1} = \bar{I}_i^{i+1} + m_i ([\mathbf{r}_{o_i c_i}^{i+1}]_{\times})^T [\mathbf{r}_{o_i c_i}^{i+1}]_{\times}$$

From the Equation (74), it is clear that the kinetic energy of the unified rigid body i can be expressed linearly with respect to the dynamic parameters, i.e. the mass m_i , three components of the first order moment of inertia $m_i \mathbf{r}_{o_i c_i}^{i+1} = [r_{ix} \ r_{iy} \ r_{iz}]^T$, six components of the symmetry inertia tensor $\bar{I}_{o_i}^{i+1}$, where

$$\bar{I}_{o_i}^{i+1} = \begin{bmatrix} \bar{I}_{o_{ixx}} & \bar{I}_{o_{ixy}} & \bar{I}_{o_{ixz}} \\ \bar{I}_{o_{ixy}} & \bar{I}_{o_{iyy}} & \bar{I}_{o_{iyz}} \\ \bar{I}_{o_{ixz}} & \bar{I}_{o_{iyz}} & \bar{I}_{o_{izz}} \end{bmatrix}$$

and the moments of inertia of all motors j attached to link i around their rotation axes $I_{m_{z_j}}$.

In a similar way, potential energy can also be expressed as a linear function of the dynamic parameters. Let $\mathbf{g}^{i+1} = (\mathbf{R}_{i+1}^0)^T \mathbf{g}^0$ denote the gravitational acceleration vector expressed in frame $i+1$. Then, the potential energy due to the gravity of the unified rigid body i can be written as:

$$\mathbb{U}_{g,i} = - \left(m_i (\mathbf{g}^{i+1})^T \mathbf{P}_i^{i+1} + \sum_{m \in \text{link } i} \left(m_{m_j} (\mathbf{g}^{i+1})^T \mathbf{P}_{m_j}^{i+1} \right) \right) \\ = -m_i (\mathbf{g}^{i+1})^T \mathbf{P}_i^{i+1} = -(\mathbf{g}^{i+1})^T (m_i \mathbf{P}_{o_i}^{i+1} + m_i \mathbf{r}_{o_i c_i}^{i+1}) \quad (75)$$

Let $\mathbf{I}_{m_z}$ denote the vector collecting moments of inertia ($I_{m_{z_j}}$) of all motors j attached to link i around their axes of rotation and let

$$\boldsymbol{\pi}_i = \begin{bmatrix} m_i & r_{ix} & r_{iy} & r_{iz} & \bar{I}_{o_{ixx}} & \bar{I}_{o_{ixy}} & \bar{I}_{o_{ixz}} & \bar{I}_{o_{iyy}} & \bar{I}_{o_{iyz}} & \bar{I}_{o_{izz}} & \mathbf{I}_{m_z}^T \end{bmatrix}^T, \quad i = 1, \dots, N \quad (76)$$

denote the vector of constant dynamic parameters. Then, by adding the kinetic energy (74) and the gravitational potential energy (75) for all unified rigid bodies and the elastic potential energy $\mathbb{U}_e$, the Lagrangian of the robotic manipulator becomes:

$$\mathbb{L} = \sum_{i=1}^N (\mathbb{T}_i - \mathbb{U}_{g,i}) - \mathbb{U}_e = \sum_{i=1}^N \left(\mathbf{Y}_{\mathbb{T}_i}^T - \mathbf{Y}_{\mathbb{U}_{g,i}}^T \right) \boldsymbol{\pi}_i - \mathbb{U}_e \quad (77)$$

where vectors $\mathbf{Y}_{\mathbb{T}_i}^T = \mathbf{Y}_{\mathbb{T}_i}^T(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})$ and $\mathbf{Y}_{\mathbb{U}_{g,i}}^T = \mathbf{Y}_{\mathbb{U}_{g,i}}^T(\mathbf{q})$ are introduced so that dynamic parameters of link i can be collected to vector $\boldsymbol{\pi}_i$:

$$\mathbb{T}_i = \mathbf{Y}_{\mathbb{T}_i}^T \boldsymbol{\pi}_i \quad (78)$$

$$\mathbb{U}_{g,i} = \mathbf{Y}_{\mathbb{U}_{g,i}}^T \boldsymbol{\pi}_i \quad (79)$$

Then, Lagrange's equations can be written as:

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathbb{L}}{\partial \mathbf{q}} = \boldsymbol{\tau}_{mr} + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \frac{d}{dt} \left(\frac{\partial \mathbb{L}}{\partial \dot{\boldsymbol{\theta}}} \right) - \frac{\partial \mathbb{L}}{\partial \boldsymbol{\theta}} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\sum_{i=1}^N \left(\left(\frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial \dot{q}_1} - \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial q_1} + \frac{\partial \mathbf{Y}_{\mathbb{U}_{g,i}}^T}{\partial q_1} \right) \boldsymbol{\pi}_i \right) \right. \\ \left. \sum_{i=1}^N \left(\left(\frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial \dot{q}_N} - \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial q_N} + \frac{\partial \mathbf{Y}_{\mathbb{U}_{g,i}}^T}{\partial q_N} \right) \boldsymbol{\pi}_i \right) \right] \\ - \boldsymbol{\tau}_{elas}^q = \boldsymbol{\tau}_{mr} + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \end{cases} \quad (80)$$

$$\Leftrightarrow \begin{cases} \left[\sum_{i=1}^N \left(\frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial \dot{\theta}_1} \boldsymbol{\pi}_i \right) \right. \\ \left. \dots \right. \\ \left. \sum_{i=1}^N \left(\frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_i}^T}{\partial \dot{\theta}_{N_e}} \boldsymbol{\pi}_i \right) \right] + \boldsymbol{\tau}_{elas} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \\ \mathbf{Y}_q \boldsymbol{\pi}_q = \boldsymbol{\tau}_q + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \mathbf{Y}_{\theta} \boldsymbol{\pi}_{\theta} + \boldsymbol{\tau}_{elas} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{cases} \quad (81)$$

where $\boldsymbol{\pi}_q$ is a vector collecting all the vectors of constant dynamic parameters $\boldsymbol{\pi}_i, i = 1, \dots, N$:

$$\boldsymbol{\pi}_q = [\boldsymbol{\pi}_1^T \ \boldsymbol{\pi}_2^T \ \dots \ \boldsymbol{\pi}_N^T]^T \\ \mathbf{Y}_q = \mathbf{Y}_q(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) = \begin{bmatrix} \mathbf{Y}_{q11} & \mathbf{Y}_{q12} & \dots & \mathbf{Y}_{q1N} \\ & \vdots & & \\ \mathbf{Y}_{qN1} & \mathbf{Y}_{qN2} & \dots & \mathbf{Y}_{qNN} \end{bmatrix} \\ \mathbf{Y}_{qij} = \frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_j}^T}{\partial \dot{q}_i} - \frac{\partial \mathbf{Y}_{\mathbb{T}_j}^T}{\partial q_i} + \frac{\partial \mathbf{Y}_{\mathbb{U}_{g,i}}^T}{\partial q_i} \quad (82)$$

To derive $\mathbf{Y}_{\theta}$ and $\boldsymbol{\pi}_{\theta}$, firstly notice that the total kinetic energy of the robot can be calculated as below:

$$\mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = \sum_{j=1}^N \mathbf{Y}_{\mathbb{T}_j}^T \boldsymbol{\pi}_j \quad (83)$$

Furthermore, noticing that $\dot{\theta}_k$ only appears in the last two terms in the Equation (74). For ease of reading, in the equation below, the k^{th} elastic joint is assumed to correspond to joint i of the robotic manipulator. As a result, only the kinetic energy expression of the unified rigid body l_{m_i} ($\mathbb{T}_{l_{m_i}}$), where motor i is attached to, remains when taking time-derivative of the lower part of Equations (80). The reason for this is that $\dot{\theta}_{k,i}$ only appears in the expression of $\mathbb{T}_{l_{m_i}}$. In addition, noticing that $\mathbf{z}_{m_i}^{l_{m_i}+1}$ is a constant vector because both vector $\mathbf{z}_{m_i}^{l_{m_i}+1}$ and frame $l_{m_i}+1$ are rigidly attached to link l_{m_i} . Then, expanding the lower part of Equations (80), adopting expression (83) and the kinetic energy expression (74), we have:

$$\begin{aligned} & \sum_{j=1}^N \left(\frac{d}{dt} \frac{\partial \mathbf{Y}_{\mathbb{T}_j}^T}{\partial \dot{\theta}_{k,i}} \boldsymbol{\pi}_j \right) \\ &= \frac{d}{dt} \frac{\partial \mathbb{T}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})}{\partial \dot{\theta}_{k,i}} = \frac{d}{dt} \frac{\partial \mathbb{T}_{l_{m_i}}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})}{\partial \dot{\theta}_{k,i}} \\ &= \frac{d}{dt} \left(\eta_{m_i} (\mathbf{z}_{m_i}^{l_{m_i}+1})^T \boldsymbol{\omega}_{l_{m_i}}^{l_{m_i}+1} + \eta_{m_i}^2 \dot{\theta}_{k,i} \right) I_{m_{z_i}} \\ &= (\eta_{m_i} (\mathbf{z}_{m_i}^{l_{m_i}+1})^T \dot{\boldsymbol{\omega}}_{l_{m_i}}^{l_{m_i}+1} + \eta_{m_i}^2 \ddot{\theta}_{k,i}) I_{m_{z_i}}, \quad k = 1, \dots, N_e \\ &= \mathbf{Y}_{\theta} \boldsymbol{\pi}_{\theta} \end{aligned} \quad (84)$$

From the Equation (84), it is clear that π_θ is the vector that collects moments of inertia of the motors driving elastic joints around the axes of rotation, and

$$\begin{aligned} Y_\theta &= Y_\theta(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \ddot{\theta}) = \text{diag}(Y_{\theta_1}, Y_{\theta_2}, \dots, Y_{\theta_{N_e}}) \\ Y_{\theta_k} &= \eta_{m_i}(\mathbf{z}_{m_i}^{l_{m_i}+1})^T \dot{\omega}_{l_{m_i}}^{l_{m_i}+1} + \eta_{m_i}^2 \ddot{\theta}_{k,i} \end{aligned} \quad (85)$$

Then, combining Equations (51) and (81) gives:

$$\begin{cases} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{S}(\mathbf{q})\ddot{\theta} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\theta} + \mathbf{G}(\mathbf{q}) \\ \quad = \mathbf{Y}_q \pi_q = \tau_q + \tau_{ex} + \tau_{fr,l} \\ \mathbf{S}(\mathbf{q})^T \ddot{\mathbf{q}} + \mathbf{B}\ddot{\theta} + \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \tau_{elas} \\ \quad = \mathbf{Y}_\theta \pi_\theta + \tau_{elas} = \tau_{me} + \tau_{fr,m} \end{cases} \quad (86)$$

Equations (86) show that the dynamic model of a robotic manipulator with mixed rigid-elastic joints can be expressed linearly with respect to the dynamic parameters. This can be helpful when designing system identification algorithms.

VIII. REPRESENT DYNAMIC MATRICES AND VECTORS WITH RESPECT TO DYNAMIC PARAMETERS

In some cases, Equations (51) are used to design controllers where the errors are included in the model, e.g. adaptive controllers [35] and robust controllers. For instance, the equation of the load side can be rewritten as:

$$\begin{aligned} \mathbf{M}(\mathbf{q})\ddot{\mathbf{e}}_q + \mathbf{S}(\mathbf{q})\ddot{\theta} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{e}}_q + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\theta} + \mathbf{G}(\mathbf{q}) \\ = \mathbf{Y}_q(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{e}}_q, \ddot{\mathbf{e}}_q, \ddot{\theta})\pi_q = \tau_q + \tau_{ex} + \tau_{fr,l} \end{aligned} \quad (87)$$

where $\mathbf{q}_d$ is desired load angular position, and

$$\mathbf{e}_q = \mathbf{q}_d - \mathbf{q}$$

In these cases, to derive the regression matrix $\mathbf{Y}_q(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{e}}_q, \ddot{\mathbf{e}}_q, \ddot{\theta})$, it is convenient to represent dynamic matrices and vectors in Equation (51) ($\mathbf{M}$, $\mathbf{S}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{G}$) with respect to dynamic parameters. Firstly, the kinetic energy of the system is calculated by summing the kinetic energy of all the unified rigid bodies in Equation (74):

$$\begin{aligned} \mathbb{T} &= \sum_{i=1}^N \mathbb{T}_i \\ &= \sum_{i=1}^N \left(\frac{1}{2} m_i (\dot{\mathbf{P}}_{o_i}^{i+1})^T \dot{\mathbf{P}}_{o_i}^{i+1} \right. \\ &\quad + (\dot{\mathbf{P}}_{o_i}^{i+1})^T [\omega_i^{i+1}] \times m_i \mathbf{r}_{o_i c_i}^{i+1} + \frac{1}{2} (\omega_i^{i+1})^T \bar{I}_{o_i}^{i+1} \omega_i^{i+1} \\ &\quad + \sum_{mr \in \text{link } i} \left(\eta_{m_j} I_{m_z j} \dot{q}_j (\mathbf{z}_{m_j}^{l_{m_j}+1})^T \omega_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_z j} \dot{q}_j^2 \right) \\ &\quad \left. + \sum_{me \in \text{link } i} \left(\eta_{m_j} I_{m_z j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^{l_{m_j}+1})^T \omega_i^{i+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_z j} \dot{\theta}_{k,j}^2 \right) \right) \\ &= \sum_{i=1}^N \left(\frac{1}{2} m_i (\dot{\mathbf{P}}_{o_i}^{i+1})^T \dot{\mathbf{P}}_{o_i}^{i+1} \right. \\ &\quad \left. - (\dot{\mathbf{P}}_{o_i}^{i+1})^T [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \omega_i^{i+1} + \frac{1}{2} (\omega_i^{i+1})^T \bar{I}_{o_i}^{i+1} \omega_i^{i+1} \right) \end{aligned}$$

$$\begin{aligned} &+ \sum_r \left(\eta_{m_j} I_{m_z j} \dot{q}_j (\mathbf{z}_{m_j}^{l_{m_j}+1})^T \omega_{l_{m_j}}^{l_{m_j}+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_z j} \dot{q}_j^2 \right) \\ &+ \sum_e \left(\eta_{m_j} I_{m_z j} \dot{\theta}_{k,j} (\mathbf{z}_{m_j}^{l_{m_j}+1})^T \omega_{l_{m_j}}^{l_{m_j}+1} + \frac{1}{2} \eta_{m_j}^2 I_{m_z j} \dot{\theta}_{k,j}^2 \right) \end{aligned} \quad (88)$$

To obtain the dynamic matrices, the Jacobian matrices are introduced for the translational velocity and angular velocity:

$$\dot{\mathbf{P}}_{o_i}^{i+1} = \mathbf{J}_{vO_i}^{i+1} \dot{\mathbf{q}} \quad (89)$$

$$\omega_i^{i+1} = \mathbf{J}_{\omega O_i}^{i+1} \dot{\mathbf{q}} \quad (90)$$

where,

$$\begin{aligned} \mathbf{J}_{vO_i}^{i+1} &= (\mathbf{R}_{i+1}^0)^T \left[\frac{\partial \mathbf{P}_{o_i}^0}{\partial q_1} \quad \frac{\partial \mathbf{P}_{o_i}^0}{\partial q_2} \quad \dots \quad \frac{\partial \mathbf{P}_{o_i}^0}{\partial q_i} \quad \mathbf{0} \quad \dots \quad \mathbf{0} \right] \\ \mathbf{J}_{\omega O_i}^{i+1} &= (\mathbf{R}_{i+1}^0)^T [\rho_1 \mathbf{z}_1^0 \quad \rho_2 \mathbf{z}_2^0 \quad \dots \quad \rho_i \mathbf{z}_i^0 \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \end{aligned}$$

Substituting Equations (89) and (90) into Equation (88) gives:

$$\begin{aligned} \mathbb{T} &= \frac{1}{2} \dot{\mathbf{q}}^T \sum_{i=1}^N \left(m_i (\mathbf{J}_{vO_i}^{i+1})^T \mathbf{J}_{vO_i}^{i+1} - 2 (\mathbf{J}_{vO_i}^{i+1})^T [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \mathbf{J}_{\omega O_i}^{i+1} \right. \\ &\quad \left. + (\mathbf{J}_{\omega O_i}^{i+1})^T \bar{I}_{o_i}^{i+1} \mathbf{J}_{\omega O_i}^{i+1} \right) \dot{\mathbf{q}} \\ &\quad + \frac{1}{2} \sum_r \dot{\mathbf{q}}^T \left(2 \eta_{m_j} I_{m_z j} \dot{q}_j (\mathbf{J}_{\omega O_{l_{m_j}}}^{l_{m_j}+1})^T \mathbf{z}_{m_j}^{l_{m_j}+1} \right) \\ &\quad + \frac{1}{2} \sum_r (\eta_{m_j}^2 I_{m_z j} \dot{q}_j^2) \\ &\quad + \frac{1}{2} \sum_e \dot{\mathbf{q}}^T \left(\eta_{m_j} I_{m_z j} \dot{\theta}_{k,j} (\mathbf{J}_{\omega O_{l_{m_j}}}^{l_{m_j}+1})^T \mathbf{z}_{m_j}^{l_{m_j}+1} \right) \\ &\quad + \frac{1}{2} \sum_e (\eta_{m_j}^2 I_{m_z j} \dot{\theta}_{k,j}^2) \\ &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\theta}^T \mathbf{B} \dot{\theta} + \dot{\mathbf{q}}^T \mathbf{S}(\mathbf{q}) \dot{\theta} \end{aligned} \quad (91)$$

Notice that, $\mathbf{J}_{\omega O_i}^{i+1} = \mathbf{J}_{\omega l_i}^{i+1}$. Therefore, matrices $\mathbf{B}$ and $\mathbf{S}(\mathbf{q})$ are computed similarly as in the case of the robotic manipulator with all elastic joints. Furthermore, these matrices only contain constants of the moments of inertia of motors from the set of dynamic parameters.

The mass matrix can be decomposed into three matrices:

$$\mathbf{M}(\mathbf{q}) = \mathbf{M}_J(\mathbf{q}) + \mathbf{M}_{rmc}(\mathbf{q}) + \mathbf{M}_{rmi} \quad (92)$$

The matrix $\mathbf{M}_J(\mathbf{q})$ is defined as:

$$\begin{aligned} \mathbf{M}_J(\mathbf{q}) &= \sum_{i=1}^N \left(m_i (\mathbf{J}_{vO_i}^{i+1})^T \mathbf{J}_{vO_i}^{i+1} - (\mathbf{J}_{vO_i}^{i+1})^T [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \mathbf{J}_{\omega O_i}^{i+1} \right. \\ &\quad \left. - (\mathbf{J}_{\omega O_i}^{i+1})^T [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \mathbf{J}_{vO_i}^{i+1} + (\mathbf{J}_{\omega O_i}^{i+1})^T \bar{I}_{o_i}^{i+1} \mathbf{J}_{\omega O_i}^{i+1} \right) \end{aligned} \quad (93)$$

It collects the masses m_i , moments of inertia $m_i \mathbf{r}_{o_i c_i}^{i+1}$, and inertia tensors $\bar{I}_{o_i}^{i+1}$ of the unified rigid bodies. It also includes the Jacobian matrices of the translational velocity and angular velocity of the joints, which show the relationship between these motions and the inertia of the robotic manipulator.

The matrix $\mathbf{M}_{rmc}(\mathbf{q}) = \mathbf{M}_c(\mathbf{q}) + \mathbf{M}_c(\mathbf{q})^T$, $\mathbf{M}_c(\mathbf{q}) = [\mathbf{M}_{c_1}(\mathbf{q}) \mathbf{M}_{c_2}(\mathbf{q}) \dots \mathbf{M}_{c_N}(\mathbf{q})]$, is an $N \times N$ symmetry matrix, and the columns of the matrix $\mathbf{M}_c(\mathbf{q})$ are calculated as:

$$\mathbf{M}_{c_i}(\mathbf{q}) = \begin{cases} 0 & \text{if joint } i \text{ is an elastic joint} \\ \eta_{m_i} \mathbf{I}_{m_{z_i}} (\mathbf{J}_{\omega_{O_i}}^{l_{m_i}+1})^T \mathbf{z}_{m_i}^{l_{m_i}+1} & \text{if joint } i \text{ is a rigid joint} \end{cases} \quad (94)$$

The matrix $\mathbf{M}_{rmi} = \text{diag}(\mathbf{M}_{rmi_1}, \mathbf{M}_{rmi_2}, \dots, \mathbf{M}_{rmi_N})$ is a constant diagonal $N \times N$ matrix, where its diagonal elements are defined as:

$$\mathbf{M}_{rmi_i} = \begin{cases} 0 & \text{if joint } i \text{ is an elastic joint} \\ \eta_{m_i}^2 \mathbf{I}_{m_{z_i}} & \text{if joint } i \text{ is a rigid joint} \end{cases} \quad (95)$$

Matrices $\mathbf{M}_{rmc}(\mathbf{q})$ and $\mathbf{M}_{rmi}$ collect the inertia of motors of the rigid joints around their axes of rotation. Matrix $\mathbf{M}_{rmc}(\mathbf{q})$ shows the coupling relationship between the motion of rigid joints motor and the motion of the other joints, while the matrix $\mathbf{M}_{rmi}$ shows the inertia of the motor's own motion. Notice that inertia matrices $\mathbf{M}_{rmc}(\mathbf{q})$ and $\mathbf{M}_{rmi}$ for the rigid joints respectively correspond to matrices $\mathbf{S}(\mathbf{q})$ and $\mathbf{B}$ for the elastic joints.

From the decomposition of matrix $\mathbf{M}(\mathbf{q})$, the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ can also be decomposed as follow:

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{C}_{rmc}(\mathbf{q}, \dot{\mathbf{q}}) \quad (96)$$

where

$$\mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}_J(\mathbf{q}) - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_J(\mathbf{q})}{\partial q_1} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_J(\mathbf{q})}{\partial q_2} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_J(\mathbf{q})}{\partial q_N} \end{bmatrix}$$

and

$$\mathbf{C}_{rmc}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}_{rmc}(\mathbf{q}) - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_{rmc}(\mathbf{q})}{\partial q_1} \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_{rmc}(\mathbf{q})}{\partial q_2} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_{rmc}(\mathbf{q})}{\partial q_N} \end{bmatrix}$$

To calculate the gravity term using the Jacobian matrices, first, rewrite Equation (75):

$$\mathbb{U}_{g,i} = -m_i(\mathbf{g}^0)^T \mathbf{P}_{c_i}^0 \quad (97)$$

Then, take partial derivative with respect to $\mathbf{q}$ of Equation (73) and retrieve the vector $\dot{\mathbf{q}}$:

$$\dot{\mathbf{P}}_{c_i}^0 = \dot{\mathbf{P}}_{o_i}^0 + [\boldsymbol{\omega}_i^0] \times \mathbf{r}_{o_i c_i}^0$$

$$\begin{aligned} \Leftrightarrow \frac{\partial \mathbf{P}_{c_i}^0}{\partial \mathbf{q}} \dot{\mathbf{q}} &= \mathbf{J}_{vO_i}^0 \dot{\mathbf{q}} - [\mathbf{r}_{o_i c_i}^0] \times \mathbf{J}_{\omega O_i}^0 \dot{\mathbf{q}} \\ \Leftrightarrow \frac{\partial \mathbf{P}_{c_i}^0}{\partial \mathbf{q}} &= \mathbf{J}_{vO_i}^0 - [\mathbf{r}_{o_i c_i}^0] \times \mathbf{J}_{\omega O_i}^0 \end{aligned} \quad (98)$$

Take the derivative of $\mathbb{U}_{g,i}$ with respect to $\mathbf{q}$:

$$\begin{aligned} \frac{\partial \mathbb{U}_{g,i}}{\partial \mathbf{q}} &= -m_i(\mathbf{g}^0)^T \frac{\partial \mathbf{P}_{c_i}^0}{\partial \mathbf{q}} = -m_i(\mathbf{g}^0)^T (\mathbf{J}_{vO_i}^0 - [\mathbf{r}_{o_i c_i}^0] \times \mathbf{J}_{\omega O_i}^0) \\ &= -(\mathbf{g}^{i+1})^T (m_i \mathbf{J}_{vO_i}^{i+1} - [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \mathbf{J}_{\omega O_i}^{i+1}) \end{aligned} \quad (99)$$

Then, the gravitational term is:

$$\mathbf{G}(\mathbf{q}) = \sum_{i=1}^N \frac{\partial \mathbb{U}_{g,i}}{\partial \mathbf{q}} \quad (100)$$

It is clear that the gravitational term can be expressed as a linear function of the mass and the moments of inertia from the set of dynamic parameters.

Now, substituting Equations (92) and (96) into Equations (51) gives:

$$\begin{cases} \mathbf{M}_J(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{M}_{rmi}\ddot{\mathbf{q}} + \mathbf{M}_{rmc}(\mathbf{q})\ddot{\mathbf{q}} \\ + \mathbf{C}_{rmc}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{S}(\mathbf{q})\ddot{\boldsymbol{\theta}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}_q + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \mathbf{S}(\mathbf{q})^T \ddot{\mathbf{q}} + \mathbf{B}\ddot{\boldsymbol{\theta}} + \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \boldsymbol{\tau}_{elas} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{cases} \quad (101)$$

The upper equation in (101) shows that the load side dynamics contains three different parts. The first part is $\mathbf{M}_J(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q})$, representing the dynamics of the robotic manipulator due to its mass, moment of inertia, and inertia tensor. The second part is $\mathbf{M}_{rmi}\ddot{\mathbf{q}} + \mathbf{M}_{rmc}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_{rmc}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$, representing the dynamics of the motors of rigid joints. The last part is $\mathbf{S}(\mathbf{q})\ddot{\boldsymbol{\theta}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}}$, representing the coupling dynamics between the load side and motor side. As a result, the regression matrix of the load side can also be decomposed into three parts:

$$\begin{aligned} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{S}(\mathbf{q})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\mathbf{q}) \\ = \mathbf{Y}_q \boldsymbol{\pi}_q = \mathbf{Y}_J \boldsymbol{\pi}_J + \mathbf{Y}_{mr} \boldsymbol{\pi}_{mr} + \mathbf{Y}_c \boldsymbol{\pi}_\theta \end{aligned} \quad (102)$$

where

$$\begin{aligned} \mathbf{Y}_J \boldsymbol{\pi}_J &= \mathbf{M}_J(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) \\ \mathbf{Y}_{mr} \boldsymbol{\pi}_{mr} &= \mathbf{M}_{rmi}\ddot{\mathbf{q}} + \mathbf{M}_{rmc}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_{rmc}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \\ \mathbf{Y}_c \boldsymbol{\pi}_\theta &= \mathbf{S}(\mathbf{q})\ddot{\boldsymbol{\theta}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} \end{aligned}$$

Vector $\boldsymbol{\pi}_J$ consists of mass, moment of inertia, and inertia tensor of all unified rigid bodies, vector $\boldsymbol{\pi}_{mr}$ consists of inertia of motors of the rigid joints around their axes of rotation, and vector $\boldsymbol{\pi}_\theta$ consists of inertia of motors of the elastic joints around their axes of rotation.

IX. MODELLING ROBOTIC MANIPULATORS WITH MIXED RIGID-ELASTIC JOINTS USING RECURSIVE NEWTON-EULER ALGORITHM

A. FORWARD KINEMATICS

In the forward kinematics, velocities and accelerations are recursively propagated from the base link 0 to the end-effector. Firstly, if joint i is a revolute joint, the angular velocity of link

i is the sum of the angular velocity of link $i-1$ and the angular velocity of joint i . Thus:

$$\omega_i^0 = \omega_{i-1}^0 + \dot{q}_i z_i^0 = \omega_{i-1}^0 + \dot{q}_i R_i^0 z_0 \quad (103)$$

where R_i^0 is the 3×3 rotation matrix from frame i to frame 0, and $z_0 = [0 \ 0 \ 1]^T$.

If joint i is a prismatic joint, the angular velocity of link i is also the angular velocity of link $i-1$. Hence:

$$\omega_i^0 = \omega_{i-1}^0 \quad (104)$$

The angular acceleration of link i can be derived by taking the time derivative of ω_i^0 . Let $[a]_\times$ denote the skew-symmetric matrix corresponding to an arbitrary vector a . If joint i is a revolute joint, then:

$$\dot{\omega}_i^0 = \dot{\omega}_{i-1}^0 + \ddot{q}_i z_i^0 + \dot{q}_i \dot{R}_i^0 z_0 = \dot{\omega}_{i-1}^0 + \ddot{q}_i z_i^0 + \dot{q}_i [\omega_{i-1}^0]_\times z_i^0 \quad (105)$$

and if joint i is a prismatic joint, then:

$$\dot{\omega}_i^0 = \dot{\omega}_{i-1}^0 \quad (106)$$

To reduce computational effort, ω_i^0 and $\dot{\omega}_i^0$ can be expressed in frame $i+1$ rigidly attached to link i :

$$\omega_i^{i+1} = \begin{cases} (R_{i+1}^i)^T (\omega_{i-1}^0 + \dot{q}_i z_0) & \text{Revolute joint } i \\ (R_{i+1}^i)^T \omega_{i-1}^0 & \text{Prismatic joint } i \end{cases} \quad (107)$$

$$\dot{\omega}_i^{i+1} = \begin{cases} (R_{i+1}^i)^T (\dot{\omega}_{i-1}^0 + \ddot{q}_i z_0 + \dot{q}_i [\omega_{i-1}^0]_\times z_0) & \text{Revolute joint } i \\ (R_{i+1}^i)^T \dot{\omega}_{i-1}^0 & \text{Prismatic joint } i \end{cases} \quad (108)$$

The angular velocities and accelerations of the motors are derived depending on the type of joints they are driving. For motor j attached to link i , if joint j , driven by motor j , is the k^{th} elastic joint, then:

$$\omega_{m_j}^0 = \omega_i^0 + \eta_{m_j} \dot{\theta}_{k,j} z_{m_j}^0 = \omega_i^0 + \eta_{m_j} \dot{\theta}_{k,j} R_{i+1}^0 z_{m_j}^{i+1} \quad (109)$$

Notice that the vector $z_{m_j}^{i+1}$ is a constant vector because both motor j and frame $i+1$ are rigidly attached to link i . The angular acceleration of the motor j is obtained by taking the time derivative of $\omega_{m_j}^0$:

$$\begin{aligned} \dot{\omega}_{m_j}^0 &= \dot{\omega}_i^0 + \eta_{m_j} \ddot{\theta}_{k,j} z_{m_j}^0 + \eta_{m_j} \dot{\theta}_{k,j} \dot{R}_{i+1}^0 z_{m_j}^{i+1} \\ &= \dot{\omega}_i^0 + \eta_{m_j} \ddot{\theta}_{k,j} z_{m_j}^0 + \eta_{m_j} \dot{\theta}_{k,j} [\omega_i^0]_\times z_{m_j}^0 \end{aligned} \quad (110)$$

Expressing $\dot{\omega}_{m_j}^0$ in frame $i+1$ gives:

$$\dot{\omega}_{m_j}^{i+1} = \dot{\omega}_i^{i+1} + \eta_{m_j} \ddot{\theta}_{k,j} z_{m_j}^{i+1} + \eta_{m_j} \dot{\theta}_{k,j} [\omega_i^{i+1}]_\times z_{m_j}^{i+1} \quad (111)$$

Similarly, for motor j attached to link i , if joint j , driven by motor j , is a rigid joint, then the angular acceleration of motor j is written as:

$$\dot{\omega}_{m_j}^{i+1} = \dot{\omega}_i^{i+1} + \eta_{m_j} \ddot{q}_j z_{m_j}^{i+1} + \eta_{m_j} \dot{q}_j [\omega_i^{i+1}]_\times z_{m_j}^{i+1} \quad (112)$$

Let $P_{o_i}^0$ denote the position vector of the origin of frame i . Let $r_{o_{i-1},i}^0$ denote the vector pointing from $P_{o_{i-1}}^0$ to $P_{o_i}^0$

expressed in the base frame. For deriving the linear acceleration, firstly, the position of the origin of frame i is represented according to the position of the origin of frame $i-1$:

$$P_{o_i}^0 = P_{o_{i-1}}^0 + r_{o_{i-1},i}^0 = P_{o_{i-1}}^0 + R_i^0 r_{o_{i-1},i}^i \quad (113)$$

The linear velocity of the origin of frame i is obtained by taking the time derivative of $P_{o_i}^0$. Notice that, in the case of revolute joints, $r_{o_{i-1},i}^i$ is a constant vector. If joint $i-1$ is a revolute joint, then:

$$\dot{P}_{o_i}^0 = \dot{P}_{o_{i-1}}^0 + \dot{R}_i^0 r_{o_{i-1},i}^i = \dot{P}_{o_{i-1}}^0 + [\omega_{i-1}^0]_\times R_i^0 r_{o_{i-1},i}^i \quad (114)$$

and if joint $i-1$ is a prismatic joint, then:

$$\begin{aligned} \dot{P}_{o_i}^0 &= \dot{P}_{o_{i-1}}^0 + \dot{R}_i^0 r_{o_{i-1},i}^i + R_i^0 \dot{r}_{o_{i-1},i}^i \\ &= \dot{P}_{o_{i-1}}^0 + [\omega_{i-1}^0]_\times R_i^0 r_{o_{i-1},i}^i + R_i^0 \dot{q}_{i-1} z_{i-1}^i \end{aligned} \quad (115)$$

where $\dot{r}_{o_{i-1},i}^i$ is the relative velocity between the origin of frame $i-1$ and the origin of frame i , represented in frame i , and $\dot{r}_{o_{i-1},i}^i = \dot{q}_{i-1} z_{i-1}^i$. Notice that, with prismatic joints, vector z_{i-1}^i is constant.

Then, the linear acceleration of the origin of frame i is obtained by taking the time derivative of $\dot{P}_{o_i}^0$. If joint $i-1$ is a revolute joint, then:

$$\begin{aligned} \ddot{P}_{o_i}^0 &= \ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^0 + [\omega_{i-1}^0]_\times (\dot{R}_i^0 r_{o_{i-1},i}^i) \\ &= \ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^0 + [\omega_{i-1}^0]_\times ([\omega_{i-1}^0]_\times r_{o_{i-1},i}^0) \end{aligned} \quad (116)$$

and if joint $i-1$ is a prismatic joint, then:

$$\begin{aligned} \ddot{P}_{o_i}^0 &= \ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^0 + [\omega_{i-1}^0]_\times (\dot{R}_i^0 r_{o_{i-1},i}^i) \\ &\quad + [\omega_{i-1}^0]_\times (R_i^0 \dot{r}_{o_{i-1},i}^i) + \dot{R}_i^0 \dot{q}_{i-1} z_{i-1}^i + R_i^0 \ddot{q}_{i-1} z_{i-1}^i \\ &= \ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^0 + [\omega_{i-1}^0]_\times ([\omega_{i-1}^0]_\times r_{o_{i-1},i}^0) \\ &\quad + 2[\omega_{i-1}^0]_\times R_i^0 \dot{q}_{i-1} z_{i-1}^i + R_i^0 \ddot{q}_{i-1} z_{i-1}^i \end{aligned} \quad (117)$$

The linear acceleration of the centre of mass of the unified rigid body i is obtained by taking the time derivative from Equation (72):

$$\ddot{P}_{c_i}^0 = \ddot{P}_{o_i}^0 + [\dot{\omega}_i^0]_\times r_{o_i c_i}^0 + [\omega_i^0]_\times ([\omega_i^0]_\times r_{o_i c_i}^0) \quad (118)$$

Expressing $\ddot{P}_{o_i}^0$ and $\ddot{P}_{c_i}^0$ in frame $i+1$ gives:

$$\ddot{P}_{o_i}^{i+1} = \begin{cases} (R_{i+1}^i)^T \left(\ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^i \right) & \text{Revolute joint } i \\ (R_{i+1}^i)^T \left(\ddot{P}_{o_{i-1}}^0 + [\dot{\omega}_{i-1}^0]_\times r_{o_{i-1},i}^i \right) & \text{Prismatic joint } i \\ + [\omega_{i-1}^0]_\times ([\omega_{i-1}^0]_\times r_{o_{i-1},i}^i) \\ + 2[\omega_{i-1}^0]_\times \dot{q}_{i-1} z_{i-1}^i + \ddot{q}_{i-1} z_{i-1}^i \end{cases} \quad (119)$$

and

$$\ddot{P}_{c_i}^{i+1} = \ddot{P}_{o_i}^{i+1} + [\dot{\omega}_i^{i+1}]_\times r_{o_i c_i}^{i+1} + [\omega_i^{i+1}]_\times ([\omega_i^{i+1}]_\times r_{o_i c_i}^{i+1}) \quad (120)$$

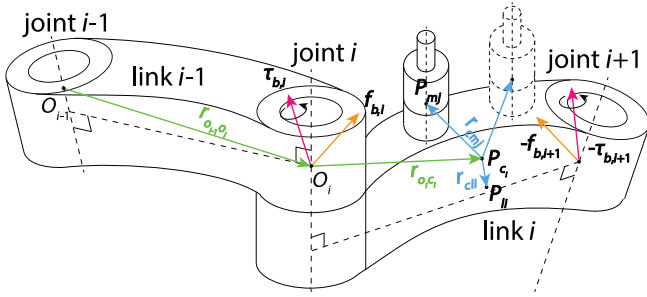


FIGURE 6: The diagram describes the force/torques acting on link i .

B. BACKWARD RECURSION

In the backward recursion, forces and torques are propagated from the end-effector to the base link. Notice that, the motors attached to the links have interaction forces with the links. However, by adopting the unified rigid bodies, these internal interaction forces between the motors and links are cancelled. Figure 6 illustrates the forces/torques acting on link i . Let $\mathbf{f}_{b,i}^0$ denote the force applied from unified rigid body $i-1$ to unified rigid body i . As a result, the force applied from unified rigid body $i+1$ to unified rigid body i is $-\mathbf{f}_{b,i+1}^0$. Then, Newton's equation for the translational motion of the unified rigid body i is:

$$\begin{aligned} \mathbf{f}_{b,i}^0 - \mathbf{f}_{b,i+1}^0 + m_i \mathbf{g}^0 &= m_i \ddot{\mathbf{P}}_{o_i}^0 \\ \Leftrightarrow \mathbf{f}_{b,i}^{i+1} &= \mathbf{R}_{i+2}^{i+1} \mathbf{f}_{b,i+1}^{i+2} + m_i (\ddot{\mathbf{P}}_{o_i}^{i+1} - \mathbf{g}^{i+1}) \end{aligned} \quad (121)$$

Let $\mathbf{f}_e^{n+1}$ denote the external force applied to the end-effector expressed in frame $n+1$. Then the initial force for the backward recursion is:

$$\mathbf{f}_{b,n+1}^{n+1} = \mathbf{f}_e^{n+1} \quad (122)$$

Let $\mathbf{r}_{o_{i+1}c_i}^0$ denote the vector pointing from $\mathbf{P}_{o_{i+1}}^0$ to $\mathbf{P}_{c_i}^0$ and $\mathbf{r}_{o_{i+1}c_i}^0 = \mathbf{P}_{c_i}^0 - \mathbf{P}_{o_{i+1}}^0$. Let $\boldsymbol{\tau}_{b,i}^0$ denote the torque applied from unified rigid body $i-1$ to unified rigid body i . As a result, the torque applied from unified rigid body $i+1$ to unified rigid body i is $-\boldsymbol{\tau}_{b,i+1}^0$. Note that the momentum of the unified rigid body i around its centre of mass is the sum of the momentum of link i and the momentum of the motors attached to link i due to their own rotation. Then, by applying Equation (69), Euler's equation for the rotational motion of the unified rigid body i is:

$$\begin{aligned} \boldsymbol{\tau}_{b,i}^0 - \boldsymbol{\tau}_{b,i+1}^0 + [\mathbf{f}_{b,i}^0] \times \mathbf{r}_{o_i c_i}^0 - [\mathbf{f}_{b,i+1}^0] \times \mathbf{r}_{o_{i+1} c_i}^0 \\ = \frac{d}{dt} \left(\bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0 + \sum_{mr \in \text{link } i} \eta_{m_j} \dot{q}_j I_{m_z j} \mathbf{z}_{m_j}^0 \right. \\ \left. + \sum_{me \in \text{link } i} \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^0 \right) \end{aligned} \quad (123)$$

Notice that, when expressed in frame $i+1$, the inertia tensor of the unified rigid body i (i.e. $\bar{\mathbf{I}}_i^{i+1}$) becomes a constant tensor. Furthermore, the relationship between the inertia tensor expressed in frame 0 and frame $i+1$ is $\bar{\mathbf{I}}_i^0 = \mathbf{R}_{i+1}^0 \bar{\mathbf{I}}_i^{i+1} (\mathbf{R}_{i+1}^0)^T$.

Then, considering the first term on the right-hand side of Equation (123):

$$\begin{aligned} \frac{d}{dt} (\bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0) \\ = \frac{d}{dt} (\mathbf{R}_{i+1}^0 \bar{\mathbf{I}}_i^{i+1} (\mathbf{R}_{i+1}^0)^T \boldsymbol{\omega}_i^0) \\ = \bar{\mathbf{I}}_i^0 \dot{\boldsymbol{\omega}}_i^0 + \dot{\mathbf{R}}_{i+1}^0 \bar{\mathbf{I}}_i^{i+1} (\mathbf{R}_{i+1}^0)^T \boldsymbol{\omega}_i^0 + \mathbf{R}_{i+1}^0 \bar{\mathbf{I}}_i^{i+1} (\dot{\mathbf{R}}_{i+1}^0)^T \boldsymbol{\omega}_i^0 \\ = \bar{\mathbf{I}}_i^0 \dot{\boldsymbol{\omega}}_i^0 + [\boldsymbol{\omega}_i^0] \times (\bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0) + \bar{\mathbf{I}}_i^0 [\boldsymbol{\omega}_i^0] \times \boldsymbol{\omega}_i^0 \\ = \bar{\mathbf{I}}_i^0 \dot{\boldsymbol{\omega}}_i^0 + [\boldsymbol{\omega}_i^0] \times (\bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0) \end{aligned} \quad (124)$$

Considering the last two terms on the right-hand side of Equation (123):

$$\begin{aligned} \frac{d}{dt} \left(\sum_{mr \in \text{link } i} \eta_{m_j} \dot{q}_j I_{m_z j} \mathbf{z}_{m_j}^0 + \sum_{me \in \text{link } i} \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^0 \right) \\ = \frac{d}{dt} \left(\sum_{mr \in \text{link } i} \eta_{m_j} \dot{q}_j I_{m_z j} \mathbf{R}_{i+1}^0 \mathbf{z}_{m_j}^{i+1} \right. \\ \left. + \sum_{me \in \text{link } i} \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} \mathbf{R}_{i+1}^0 \mathbf{z}_{m_j}^{i+1} \right) \\ = \sum_{mr \in \text{link } i} \left(\eta_{m_j} \ddot{q}_j I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{q}_j I_{m_z j} \dot{\mathbf{R}}_{i+1}^0 \mathbf{z}_{m_j}^{i+1} \right) \\ + \sum_{me \in \text{link } i} \left(\eta_{m_j} \ddot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} \dot{\mathbf{R}}_{i+1}^0 \mathbf{z}_{m_j}^{i+1} \right) \\ = \sum_{mr \in \text{link } i} \left(\eta_{m_j} \ddot{q}_j I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{q}_j I_{m_z j} [\boldsymbol{\omega}_i^0] \times \mathbf{z}_{m_j}^0 \right) \\ + \sum_{me \in \text{link } i} \left(\eta_{m_j} \ddot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} [\boldsymbol{\omega}_i^0] \times \mathbf{z}_{m_j}^0 \right) \end{aligned} \quad (125)$$

Substituting Equations (124) and (125) into Equation (123) yields:

$$\begin{aligned} \boldsymbol{\tau}_{b,i}^0 &= \boldsymbol{\tau}_{b,i+1}^0 - [\mathbf{f}_{b,i}^0] \times \mathbf{r}_{o_i c_i}^0 + [\mathbf{f}_{b,i+1}^0] \times \mathbf{r}_{o_{i+1} c_i}^0 \\ &+ \bar{\mathbf{I}}_i^0 \dot{\boldsymbol{\omega}}_i^0 + [\boldsymbol{\omega}_i^0] \times (\bar{\mathbf{I}}_i^0 \boldsymbol{\omega}_i^0) \\ &+ \sum_{mr \in \text{link } i} \left(\eta_{m_j} \ddot{q}_j I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{q}_j I_{m_z j} [\boldsymbol{\omega}_i^0] \times \mathbf{z}_{m_j}^0 \right) \\ &+ \sum_{me \in \text{link } i} \left(\eta_{m_j} \ddot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^0 + \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} [\boldsymbol{\omega}_i^0] \times \mathbf{z}_{m_j}^0 \right) \end{aligned} \quad (126)$$

Expressing $\boldsymbol{\tau}_{b,i}^0$ in frame $i+1$ gives:

$$\begin{aligned} \boldsymbol{\tau}_{b,i}^{i+1} &= \mathbf{R}_{i+2}^{i+1} \boldsymbol{\tau}_{b,i+1}^{i+2} - [\mathbf{f}_{b,i}^{i+1}] \times \mathbf{r}_{o_i c_i}^{i+1} + [\mathbf{R}_{i+2}^{i+1} \mathbf{f}_{b,i+1}^{i+2}] \times \mathbf{r}_{o_{i+1} c_i}^{i+1} \\ &+ \bar{\mathbf{I}}_i^{i+1} \dot{\boldsymbol{\omega}}_i^{i+1} + [\boldsymbol{\omega}_i^{i+1}] \times (\bar{\mathbf{I}}_i^{i+1} \boldsymbol{\omega}_i^{i+1}) \\ &+ \sum_{mr \in \text{link } i} \left(\eta_{m_j} \ddot{q}_j I_{m_z j} \mathbf{z}_{m_j}^{i+1} + \eta_{m_j} \dot{q}_j I_{m_z j} [\boldsymbol{\omega}_i^{i+1}] \times \mathbf{z}_{m_j}^{i+1} \right) \\ &+ \sum_{me \in \text{link } i} \left(\eta_{m_j} \ddot{\theta}_{k,j} I_{m_z j} \mathbf{z}_{m_j}^{i+1} + \eta_{m_j} \dot{\theta}_{k,j} I_{m_z j} [\boldsymbol{\omega}_i^{i+1}] \times \mathbf{z}_{m_j}^{i+1} \right) \end{aligned} \quad (127)$$

Let $\boldsymbol{\tau}_e^{n+1}$ denote the external torque applied to the end-effector expressed in frame $n+1$. Then the initial torque for the backward recursion is:

$$\boldsymbol{\tau}_{b,n+1}^{n+1} = \boldsymbol{\tau}_e^{n+1} \quad (128)$$

Let $\tau = [\tau_1 \dots \tau_N]^T$ denote the vector of generalised forces. The generalised force τ_i at joint i is the sum of the moment $\tau_{b,i}^0$ projected along the axis of rotation $\mathbf{z}_i^0$ of the joint and the inertia torque of motor i . However, the moment $\tau_{b,i}^0$ is calculated according to the type of the joint. In particular, if joint i is a rigid joint, then:

$$\tau_i = \begin{cases} (\tau_{b,i}^0)^T \mathbf{z}_i^0 + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^0)^T \mathbf{z}_{m_i}^0 & \text{Revolute joint } i \\ (\mathbf{f}_{b,i}^0)^T \mathbf{z}_i^0 + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^0)^T \mathbf{z}_{m_i}^0 & \text{Prismatic joint } i \end{cases}$$

$$\Leftrightarrow \tau_i = \begin{cases} (\tau_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Revolute joint } i \\ + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}} & \\ (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Prismatic joint } i \\ + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}} & \end{cases} \quad (129)$$

On the other hand, the motor side and the load side of an elastic joint are not directly connected but through an elastic torque. As a result, if joint i is the k^{th} elastic joint, the load side dynamics is described as follows:

$$\begin{cases} (\tau_{b,i}^0)^T \mathbf{z}_i^0 = K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \text{Revolute joint } i \\ (\mathbf{f}_{b,i}^0)^T \mathbf{z}_i^0 = K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \text{Prismatic joint } i \end{cases}$$

$$\Leftrightarrow \begin{cases} (\tau_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Revolute joint } i \\ = K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \\ (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Prismatic joint } i \\ = K_{k,i} \eta_{l_i} (\theta_{k,i} - \eta_{l_i} q_i) & \end{cases} \quad (130)$$

and the motor side dynamics is described as follows:

$$\begin{aligned} \tau_i &= K_{k,i} (\theta_{k,i} - \eta_{l_i} q_i) + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^0)^T \mathbf{z}_{m_i}^0 \\ &= K_{k,i} (\theta_{k,i} - \eta_{l_i} q_i) + \eta_{m_i} I_{m_{z_i}} (\dot{\omega}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}} \end{aligned} \quad (131)$$

The Newton-Euler recursive algorithm is summarised in the Algorithm 1, where $\mathbf{f}_b = [\mathbf{f}_{b,1}^2 \dots \mathbf{f}_{b,N}^{N+1}]$ and $\tau_b = [\tau_{b,1}^2 \dots \tau_{b,N}^{N+1}]$ are the data sets collecting the interaction forces and torques. By adopting Equations (107), (108), (111), (112), (119), (120), (121), (127), (129), (130), and (131), for given positions, velocities, and accelerations at a specific time, the reaction forces and torques at the joints and the generalised torques are computed.

X. NEWTON-EULER RECURSIVE ALGORITHM FOR COMPUTING FORWARD DYNAMICS

For simulating robotic manipulators with mixed rigid-elastic joints, forward dynamics is computed. The forward dynamics computation is to determine the joint and motor accelerations $[\ddot{\mathbf{q}}^T \ddot{\boldsymbol{\theta}}^T]^T$ given the joint and motor positions $[\mathbf{q}^T \boldsymbol{\theta}^T]^T$, velocities $[\dot{\mathbf{q}}^T \dot{\boldsymbol{\theta}}^T]^T$, and motor torques τ_m . We rewrite Equations (51) as:

$$\mathbf{A}(\mathbf{q}) \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} + \zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} \tau_q + \tau_{fr,l} \\ \tau_{me} + \tau_{fr,m} \end{bmatrix} \quad (132)$$

where

$$\mathbf{A}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix}$$

TABLE 1: Dynamic Parameters

	link 1	link 2
mass(kg)	2.5	1.5
inertia	$\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 0.35 & 0.15 & 0.1 \\ 0.15 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$
	link 3	motor 1, 2, 3
mass(kg)	1.0	1.0
inertia	$\begin{bmatrix} 0.3 & 0.15 & 0.13 \\ 0.15 & 0.25 & 0.2 \\ 0.13 & 0.2 & 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.006 & 0 & 0 \\ 0 & 0.006 & 0 \\ 0 & 0 & 0.004 \end{bmatrix}$

$$\zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}}) \dot{\boldsymbol{\theta}} \\ \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) - \tau_{ex} \\ \tau_{elas} \end{bmatrix}$$

The $(N + N_e) \times 1$ vector $\zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = [\zeta_1 \dots \zeta_{N+N_e}]^T$ can be determined by firstly running the Newton-Euler recursive algorithm and setting $[\ddot{\mathbf{q}}^T \ddot{\boldsymbol{\theta}}^T]^T = \mathbf{0}^T$. For the first N elements of vector $\zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$:

$$\zeta_i = \begin{cases} (\tau_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Elastic, revolute joint } i \\ (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Elastic, prismatic joint } i \\ \tau_i & \text{Rigid joint } i \end{cases} \quad (133)$$

and for the last N_e elements of vector $\zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, if joint i is the k^{th} elastic joint, then:

$$\zeta_{N+k} = \tau_i \quad (134)$$

The $(N + N_e) \times (N + N_e)$ mass matrix $\mathbf{A}(\mathbf{q})$ can be constructed in columns. To construct column j of matrix $\mathbf{A}(\mathbf{q})$, the Newton-Euler recursive algorithm is executed, where $[\dot{\mathbf{q}}^T \dot{\boldsymbol{\theta}}^T]^T = \mathbf{0}^T$, $\mathbf{f}_e^{n+1} = \mathbf{0}^T$, $\tau_e^{n+1} = \mathbf{0}^T$, $\mathbf{g}^0 = \mathbf{0}^T$, $[\ddot{\mathbf{q}}^T \ddot{\boldsymbol{\theta}}^T]^T = \mathbf{I}_j^T$, and $\mathbf{I}_j$ is the j^{th} column of an $(N + N_e) \times (N + N_e)$ identity matrix. Then, for the first N elements of column j of matrix $\mathbf{A}(\mathbf{q})$:

$$A_{i,j} = \begin{cases} (\tau_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Elastic, revolute joint } i \\ (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 & \text{Elastic, prismatic joint } i \\ \tau_i & \text{Rigid joint } i \end{cases} \quad (135)$$

and for the last N_e elements of column $\mathbf{A}_j$, if joint i is the k^{th} elastic joint, then:

$$A_{N+k,j} = \tau_i - K_{k,i} (\theta_{k,i} - \eta_{l_i} q_i) \quad (136)$$

Finally, the accelerations are calculated as follows:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} = \mathbf{A}(\mathbf{q})^{-1} \left(\begin{bmatrix} \tau_q + \tau_{fr,l} \\ \tau_{me} + \tau_{fr,m} \end{bmatrix} - \zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \right) \quad (137)$$

This process is summarised in the Algorithm 2.

It is well known that the complexity of the Newton-Euler method is normally less than that of the Lagrangian method. The complexity of the Newton-Euler recursive algorithm in this work is $O(n)$. To reduce the complexity of the Lagrangian method, the matrices can be precalculated and stored in symbolic form. This helps reduce the complexity of the Lagrangian method to $O(n^2)$.

Algorithm 1 Newton-Euler recursive algorithm.**function** NEREAL

Input: $[\mathbf{q}^T \ \dot{\mathbf{q}}^T \ \ddot{\mathbf{q}}^T]^T$, $[\dot{\mathbf{q}}^T \ \ddot{\mathbf{q}}^T]^T$, $[\ddot{\mathbf{q}}^T \ \ddot{\mathbf{\theta}}^T]^T$, $\mathbf{f}_e^{n+1}$, $\boldsymbol{\tau}_e^{n+1}$, $\mathbf{g}^0$, $\boldsymbol{\omega}_0^0$, $\dot{\boldsymbol{\omega}}_0^0$, $\ddot{\mathbf{p}}_{o_0}^0$

Output: $\boldsymbol{\tau}$, $\mathbf{f}_b$, $\boldsymbol{\tau}_b$

FORWARD KINEMATICS

for $i \leftarrow 1$ to N **do**

if joint i is revolute joint **then**

$$\boldsymbol{\omega}_i^{i+1} = (\mathbf{R}_{i+1}^i)^T (\boldsymbol{\omega}_{i-1}^i + \dot{q}_i \mathbf{z}_0)$$

$$\dot{\boldsymbol{\omega}}_i^{i+1} = (\mathbf{R}_{i+1}^i)^T (\dot{\boldsymbol{\omega}}_{i-1}^i + \dot{q}_i \mathbf{z}_0 + \dot{q}_i [\boldsymbol{\omega}_{i-1}^i] \times \mathbf{z}_0)$$

$$\ddot{\mathbf{p}}_{o_i}^{i+1} = (\mathbf{R}_{i+1}^i)^T \left(\ddot{\mathbf{p}}_{o_{i-1}}^i + [\dot{\boldsymbol{\omega}}_{i-1}^i] \times \mathbf{r}_{o_{i-1},i}^i + [\boldsymbol{\omega}_{i-1}^i] \times ([\boldsymbol{\omega}_{i-1}^i] \times \mathbf{r}_{o_{i-1},i}^i) \right)$$

else if joint i is prismatic joint **then**

$$\boldsymbol{\omega}_i^{i+1} = (\mathbf{R}_{i+1}^i)^T \boldsymbol{\omega}_{i-1}^i$$

$$\dot{\boldsymbol{\omega}}_i^{i+1} = (\mathbf{R}_{i+1}^i)^T \dot{\boldsymbol{\omega}}_{i-1}^i$$

$$\ddot{\mathbf{p}}_{o_i}^{i+1} = (\mathbf{R}_{i+1}^i)^T \left(\ddot{\mathbf{p}}_{o_{i-1}}^i + [\dot{\boldsymbol{\omega}}_{i-1}^i] \times \mathbf{r}_{o_{i-1},i}^i + [\boldsymbol{\omega}_{i-1}^i] \times ([\boldsymbol{\omega}_{i-1}^i] \times \mathbf{r}_{o_{i-1},i}^i) + 2[\boldsymbol{\omega}_{i-1}^i] \times \dot{q}_{i-1} \mathbf{z}_{i-1}^i + \ddot{q}_{i-1} \mathbf{z}_{i-1}^i \right)$$

end if

$$\ddot{\mathbf{p}}_{c_i}^{i+1} = \ddot{\mathbf{p}}_{o_i}^{i+1} + [\dot{\boldsymbol{\omega}}_i^{i+1}] \times \mathbf{r}_{o_i c_i}^{i+1} + [\boldsymbol{\omega}_i^{i+1}] \times ([\boldsymbol{\omega}_i^{i+1}] \times \mathbf{r}_{o_i c_i}^{i+1})$$

end for

for $i \leftarrow 1$ to N **do**

if joint i is rigid **then**

$$\dot{\boldsymbol{\omega}}_{m_i}^{l_{m_i}+1} = \dot{\boldsymbol{\omega}}_{l_{m_i}}^{l_{m_i}+1} + \eta_{m_i} \ddot{q}_i \mathbf{z}_{m_i}^{l_{m_i}+1} + \eta_{m_i} \dot{q}_i [\boldsymbol{\omega}_{l_{m_i}}^{l_{m_i}+1}] \times \mathbf{z}_{m_i}^{l_{m_i}+1}$$

else if joint i is the k^{th} elastic joint **then**

$$\dot{\boldsymbol{\omega}}_{m_i}^{l_{m_i}+1} = \dot{\boldsymbol{\omega}}_{l_{m_i}}^{l_{m_i}+1} + \eta_{m_i} \ddot{\theta}_{k,i} \mathbf{z}_{m_i}^{l_{m_i}+1} + \eta_{m_i} \dot{\theta}_{k,i} [\boldsymbol{\omega}_{l_{m_i}}^{l_{m_i}+1}] \times \mathbf{z}_{m_i}^{l_{m_i}+1}$$

end if

end for

BACKWARD RECURSION

for $i \leftarrow N$ to 1 **do**

$$\mathbf{f}_{b,i}^{i+1} = \mathbf{R}_{i+2}^{i+1} \mathbf{f}_{b,i+1}^{i+2} + m_i (\ddot{\mathbf{p}}_{c_i}^{i+1} - \mathbf{g}^{i+1})$$

$$\boldsymbol{\tau}_{b,i}^{i+1} = \mathbf{R}_{i+2}^{i+1} \boldsymbol{\tau}_{b,i+1}^{i+2} - [\mathbf{f}_{b,i}^{i+1}] \times \mathbf{r}_{o_i c_i}^{i+1}$$

$$+ [\mathbf{R}_{i+2}^{i+1} \mathbf{f}_{b,i+1}^{i+2}] \times \mathbf{r}_{o_{i+1} c_i}^{i+1} + \bar{\mathbf{l}}_i^{i+1} \dot{\boldsymbol{\omega}}_i^{i+1} + [\boldsymbol{\omega}_i^{i+1}] \times (\bar{\mathbf{l}}_i^{i+1} \dot{\boldsymbol{\omega}}_i^{i+1})$$

$$+ \sum_{m_r \in \text{link } i} \left(\eta_{m_r} \ddot{q}_r \mathbf{I}_{m_r z_j} \mathbf{z}_{m_j}^{i+1} + \eta_{m_r} \dot{q}_r \mathbf{I}_{m_r z_j} [\boldsymbol{\omega}_i^{i+1}] \times \mathbf{z}_{m_j}^{i+1} \right)$$

$$+ \sum_{m_e \in \text{link } i} \left(\eta_{m_e} \ddot{\theta}_{k,j} \mathbf{I}_{m_r z_j} \mathbf{z}_{m_j}^{i+1} + \eta_{m_e} \dot{\theta}_{k,j} \mathbf{I}_{m_r z_j} [\boldsymbol{\omega}_i^{i+1}] \times \mathbf{z}_{m_j}^{i+1} \right)$$

end for

GENERALISED TORQUE

for $i \leftarrow 1$ to N **do**

if joint i is rigid **then**

if joint i is revolute joint **then**

$$\tau_i = (\boldsymbol{\tau}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 + \eta_{m_i} \mathbf{I}_{m_i z_i} (\dot{\boldsymbol{\omega}}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}}$$

else if joint i is prismatic joint **then**

$$\tau_i = (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0 + \eta_{m_i} \mathbf{I}_{m_i z_i} (\dot{\boldsymbol{\omega}}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}}$$

end if

else if joint i is the k^{th} elastic joint **then**

$$\tau_i = K_{k,i} (\theta_{k,i} - q_i) + \eta_{m_i} \mathbf{I}_{m_i z_i} (\dot{\boldsymbol{\omega}}_{m_i}^{l_{m_i}})^T \mathbf{z}_{m_i}^{l_{m_i}}$$

end if

end for

gather components of $\boldsymbol{\tau}$, $\mathbf{f}_b$, and $\boldsymbol{\tau}_b$

return $\boldsymbol{\tau}$, $\mathbf{f}_b$, $\boldsymbol{\tau}_b$

end function

XI. THE EFFECT OF DIFFERENT JOINT STIFFNESS

In this section, the effect of different joint stiffness on robots with mixed rigid-elastic joints is investigated through simulation. The simulated robotic manipulator has three joints, as illustrated in Figure 7. The first two joints are rigid and the last one is elastic. The joint positions are denoted as q_1 , q_2 , q_3 , while the motor position of the elastic joint 3 is denoted as θ_1 . Motor 1 is fixed to the base link, whereas motors 2 and 3 are attached to link 1. The motor torques are limited to 120 (Nm). Table 1 shows the parameters of the joints and the motors used in simulations. The stiffness coefficient of joint 3 is changed between 250, 500, and 1000 (Nm/rad). The desired positions of the joints are sinusoidal

functions of the form $0.5\sin(\pi t)$ and the robotic manipulator is simulated for 5 seconds. The control gains are chosen as: $\mathbf{K}_{Pq} = \text{diag}(121, 121, 121)$ (Nm/rad), $\mathbf{K}_{Dq} = \text{diag}(22, 22, 22)$ (Nms/rad), $\mathbf{K}_{P\theta} = 1700$ (Nm/rad), and $\mathbf{K}_{D\theta} = 500$ (Nms/rad). The initial positions of the joints are:

$$[\pi/2 \ \pi/2 \ -\pi/4]^T \text{ (rad)} \quad (138)$$

Figures 8a, 8b, and 8c show the positions of the joints when the joint stiffness is 250, 500, and 1000 (Nm/rad), respectively. The joints have good tracking performance. The elastic joint has a larger overshoot when the stiffness is increased. The reaching time is quite similar in these three cases.

Figures 9a, 9b, and 9c show the positions of the motor of the elastic joint when the joint stiffness is 250, 500, and

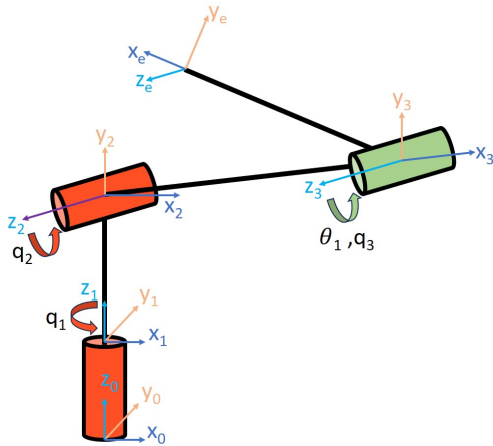
Algorithm 2 Forward dynamics calculation**function** DirDyn**Input:** $[\mathbf{q}^T \ \dot{\mathbf{q}}^T]^T, [\ddot{\mathbf{q}}^T \ \ddot{\boldsymbol{\theta}}^T]^T, \boldsymbol{\tau}_m, \mathbf{f}_e^{n+1}, \boldsymbol{\tau}_e^{n+1}$ **Output:** $[\ddot{\mathbf{q}}^T \ \ddot{\boldsymbol{\theta}}^T]^T$ **CALCULATE** $\zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ $[\boldsymbol{\tau}, \mathbf{f}_b, \boldsymbol{\tau}_b] = \text{NERREAL}([\mathbf{q}^T \ \dot{\mathbf{q}}^T]^T, [\ddot{\mathbf{q}}^T \ \ddot{\boldsymbol{\theta}}^T]^T, \mathbf{0}^T, \mathbf{f}_e^{n+1}, \boldsymbol{\tau}_e^{n+1}, \mathbf{g}^0, \boldsymbol{\omega}_0^0, \dot{\boldsymbol{\omega}}_0^0, \ddot{\mathbf{P}}_{o_0}^0)$ **for** $i \leftarrow 1$ to N **do****if** joint i is rigid **then** $\zeta_i = \tau_i$ **else if** joint i is the k^{th} elastic joint **then****if** joint i is revolute joint **then** $\zeta_i = (\boldsymbol{\tau}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0$ **else if** joint i is prismatic joint **then** $\zeta_i = (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0$ **end if** $\zeta_{N+k} = \tau_i$ **end if****end for****CALCULATE** $\mathbf{A}(\mathbf{q})$ **for** $j \leftarrow 1$ to $N + N_e$ **do** $[\boldsymbol{\tau}, \mathbf{f}_b, \boldsymbol{\tau}_b] = \text{NERREAL}([\mathbf{q}^T \ \dot{\mathbf{q}}^T]^T, \mathbf{0}^T, \mathbf{I}_j, \mathbf{0}^T, \mathbf{0}^T, \mathbf{0}^T, \boldsymbol{\omega}_0^0, \dot{\boldsymbol{\omega}}_0^0, \ddot{\mathbf{P}}_{o_0}^0)$ **for** $i \leftarrow 1$ to N **do****if** joint i is rigid **then** $A_{i,j} = \tau_i$ **else if** joint i is the k^{th} elastic joint **then****if** joint i is revolute joint **then** $A_{i,j} = (\boldsymbol{\tau}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0$ **else if** joint i is prismatic joint **then** $A_{i,j} = (\mathbf{f}_{b,i}^{i+1})^T (\mathbf{R}_{i+1}^i)^T \mathbf{z}_0$ **end if** $A_{N+k,j} = \tau_i - K_{k,i}(\theta_{k,i} - \eta_{li} q_i)$ **end if****end for****end for****CALCULATE** $[\ddot{\mathbf{q}}^T \ \ddot{\boldsymbol{\theta}}^T]^T$ Calculate $\boldsymbol{\tau}_q$ **for** $i \leftarrow 1$ to N **do****if** joint i is rigid **then** $\tau_{q_i} = \tau_{m_i}(t)$ **else if** joint i is the k^{th} elastic joint **then** $\tau_{q_i} = K_{k,i} \eta_{li} (\theta_{k,i} - \eta_{li} q_i)$ **end if****end for**Calculate $\ddot{\mathbf{q}}$ and $\ddot{\boldsymbol{\theta}}$
$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} = \mathbf{A}(\mathbf{q})^{-1} \left(\begin{bmatrix} \boldsymbol{\tau}_q + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{bmatrix} - \zeta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \right)$$
return $[\ddot{\mathbf{q}}^T \ \ddot{\boldsymbol{\theta}}^T]^T$ **end function**

FIGURE 7: Kinematic diagram of the robot used in simulation. The red joints are rigid, and the green joint is elastic. The joint positions are denoted as q_1, q_2, q_3 , respectively. The motor position of the elastic joint 3 is denoted as θ_1 .

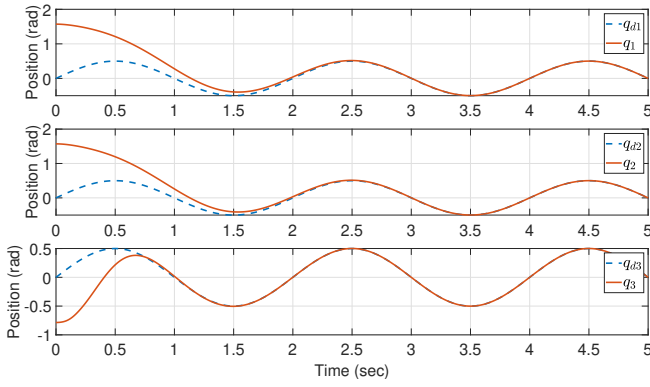
1000 (Nm/rad), respectively. The motor of the elastic joint is decoupled from the joint by a spring. Therefore, to adjust the position of the elastic joint, the deviation between the joint

and motor position is controlled so that the elastic force can move the elastic joint to the desired position. The motor of the elastic joint also has good tracking performance. When the joint stiffness increases, the desired positions for the motors of elastic joints at the beginning of the simulation are closer to the initial positions. This is because stiffer joints require less deviation between the joint position and the motor position to produce enough torque to control the robot's elastic joint.

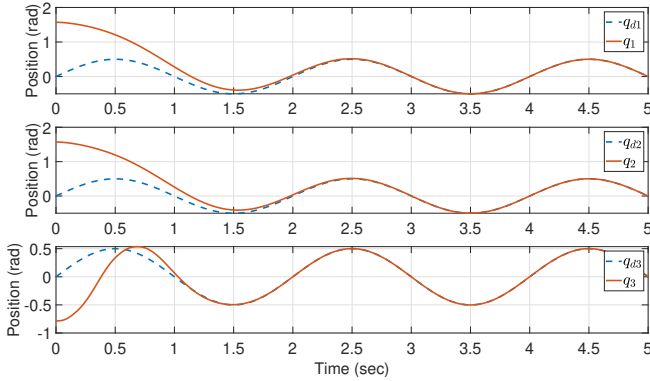
Figures 10a, 10b, and 10c show the motor torques when the joint stiffness is 250, 500, and 1000 (Nm/rad), respectively. The motor torques for the elastic joints are large at the beginning of the simulation due to the large initial position error. Furthermore, in the steady state, the elastic joints' motor torques slightly increase when the joint stiffness increases. This is because the motors need larger torques to pull the joints in the desired direction with the same deviation between the joints and the motors.

XII. THE EFFECT OF DIFFERENT ELASTIC JOINT POSITIONS

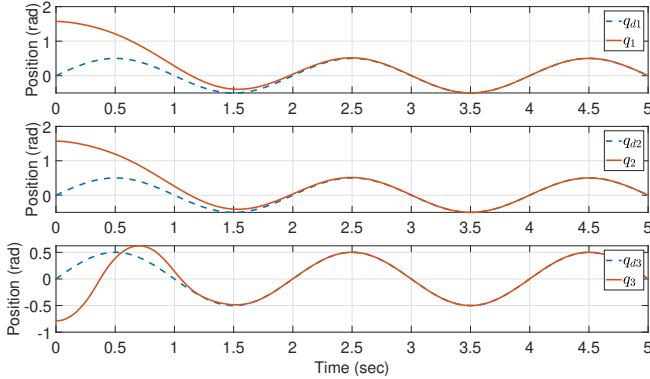
In this section, the effect of different elastic joint positions on a robot with mixed rigid-elastic joints is analysed through simulation. The robot and the control algorithm are the same as in the previous section. The robot still has 3 joints, including one elastic joint. The stiffness of the elastic joint is



(a) Joint stiffness is 250 (Nm/rad)



(b) Joint stiffness is 500 (Nm/rad)

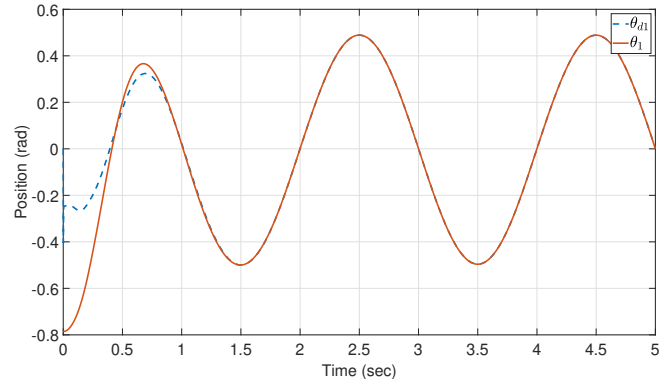


(c) Joint stiffness is 1000 (Nm/rad)

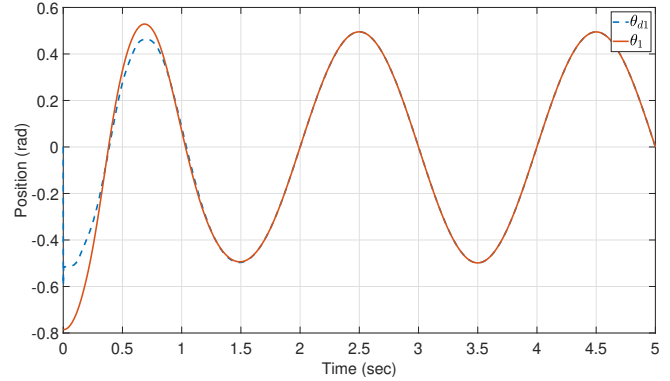
FIGURE 8: Joint coordinates $\mathbf{q}$. The dashed blue lines illustrate the desired position, and the solid red lines illustrate the actual position of the joints.

1000 (Nm/rad). Three designs of the robot are investigated: RRE, RER, and ERR, where E denotes the elastic joint and R denotes the rigid joint. In these three designs, the elastic joint is placed at joint 3 in the RRE design, joint 2 in the RER design, and joint 1 in the ERR design. The desired trajectory, the initial position of the joints, and the controller are the same as in the previous section.

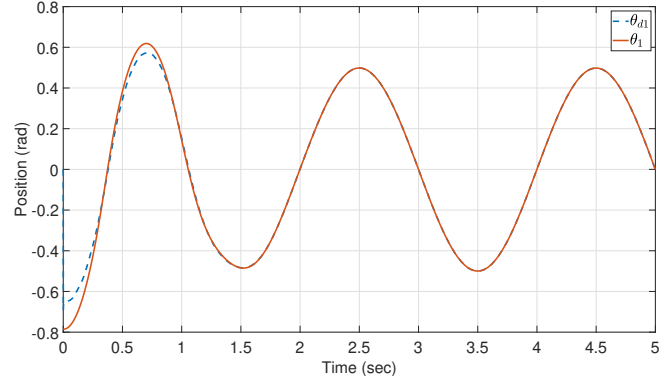
Figure 11 illustrates the end-effector in these three cases. The joint coordinates $\mathbf{q}$ and the angular positions θ of the motor of the elastic joint are shown in Figure 12 and Figure 13, respectively. It is noticeable that the elastic joint converges to



(a) Joint stiffness is 250 (Nm/rad)



(b) Joint stiffness is 500 (Nm/rad)



(c) Joint stiffness is 1000 (Nm/rad)

FIGURE 9: Angular position θ of the motor for the elastic joint. The dashed blue lines illustrate the desired position, and the solid red lines illustrate the actual position of the motor.

the desired trajectory faster than the rigid joints. This mostly depends on control algorithms. The motor of the elastic joints can follow the desired position with good precision. The plots for the motor torques, illustrated in Figure 14, show that the motor torques of the elastic joints are higher than those of the rigid joints at the beginning. This is because the elastic joint motors have to simultaneously handle two tasks: reducing tracking error and minimising elastic fluctuation. The kinetic energy, potential energy due to gravity, and total energy are illustrated in Figure 15, and the potential energy due to elasticity is illustrated in Figure 16. When the elastic joint

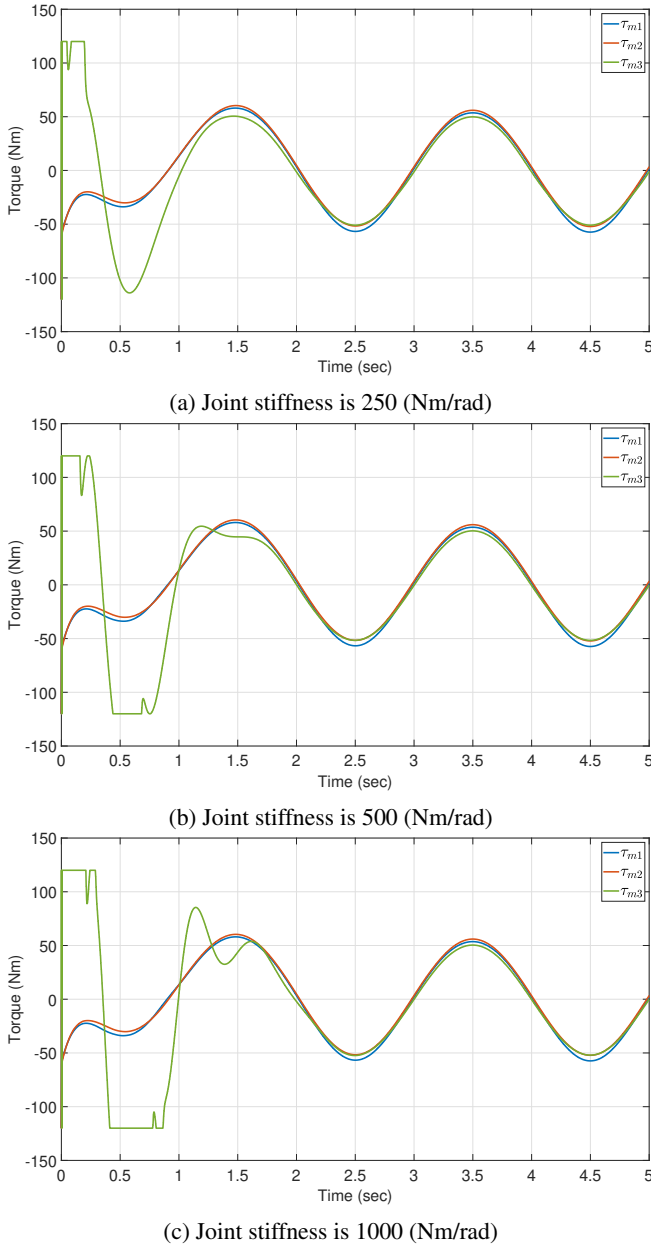


FIGURE 10: Motor torques. The green lines show the motor torques of joint 3, which is the elastic joint.

is placed closer to the base, the kinetic energy increases. In addition, the potential energy due to elasticity also increases, which leads to an increase in total energy. This implies that the robot uses more energy at the beginning to achieve similar control performance.

XIII. DESIGN HEURISTICS

In this section, the design strategies are recommended based on the analysis of the relationship between elastic joint placement, joint stiffness, and the performance of the robotic manipulator.

Figure 17 illustrates the impact of joint stiffness on the

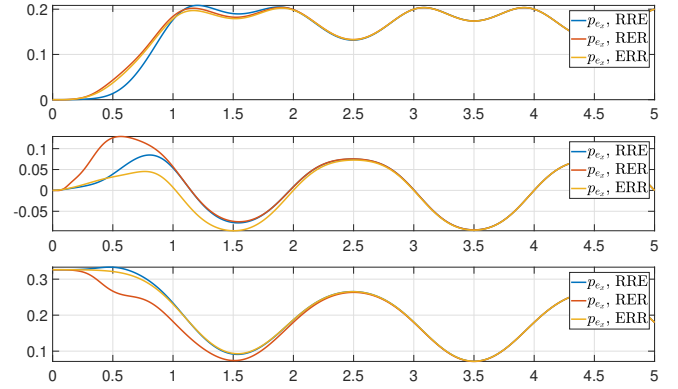


FIGURE 11: End-effector positions. The first, second, and third plots show the movement of the end-effector in the x , y , and z directions, respectively.

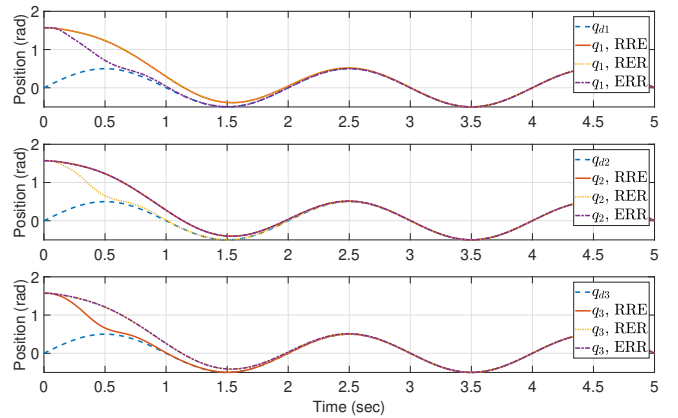
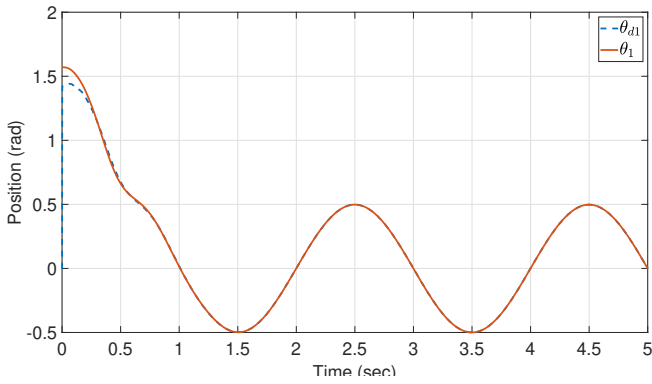


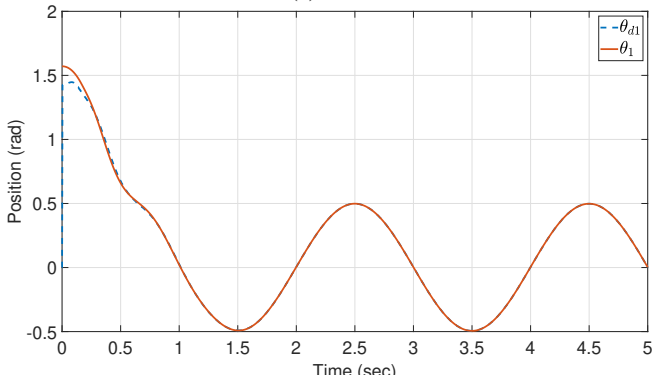
FIGURE 12: Joint coordinates $\mathbf{q}$. The dashed blue lines illustrate the desired position, and the solid lines illustrate the actual positions of the joints.

root mean squares (RMS) of the angular errors of the joints across different designs. The angular position errors of the rigid joints are not affected by the position of the elastic joints or their stiffness. On the other hand, if the stiffness is low, the RMS errors of the elastic joints are significantly higher than those of the rigid joints. As stiffness increases, the RMS errors are remarkably reduced, even lower than the errors of the rigid joints. Beyond some point (around 300 Nm/rad in the case of the considered robotic manipulator), the reduction rate slows down. Figure 18 illustrates the impact of the joint stiffness on the RMS of the motor torques in different designs. The motor torques of the rigid joints are also unaffected by the position of the elastic joints or their stiffness. Additionally, while increasing joint stiffness reduces the angular position error of the elastic joints, it also leads to higher motor torque in the elastic joints.

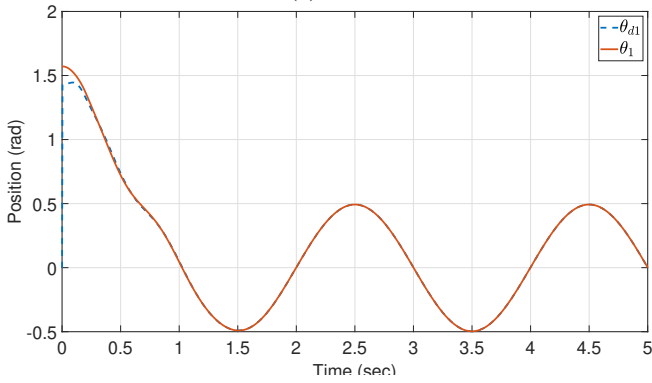
The relationship between the joint stiffness and the RMS of the potential energy due to elasticity of the robotic manipulator is illustrated in Figure 19. The closer the elastic joint is placed to the base joint, the higher the elastic potential energy is. This can be because the inertia of the links closer to the



(a) RRE



(b) RER

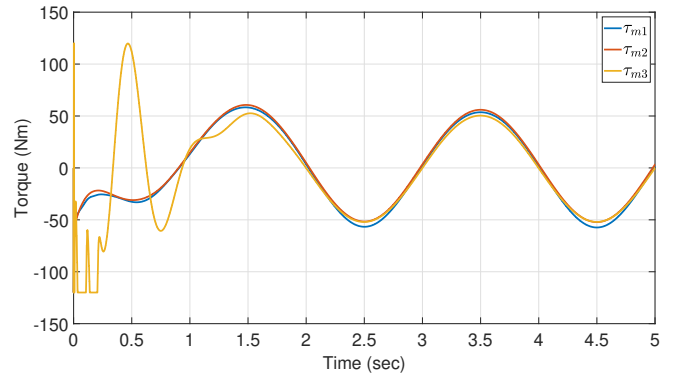


(c) ERR

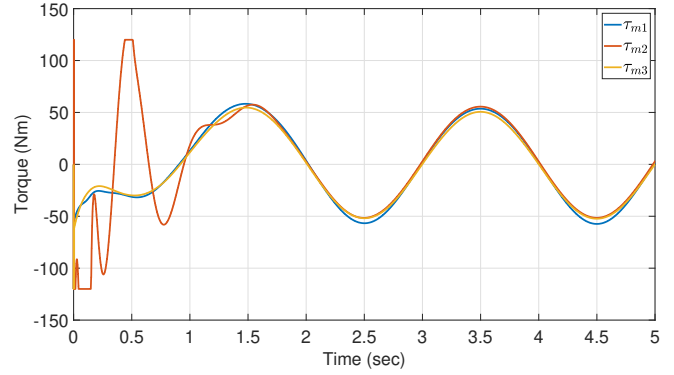
FIGURE 13: Angular position θ of the motor of the elastic joints. The dashed lines illustrate the desired position, and the solid lines illustrate the actual position of the motors.

end-effector causes more fluctuation in the elastic joints when moving. In addition, when the joint stiffness increases, the elastic potential energy is reduced.

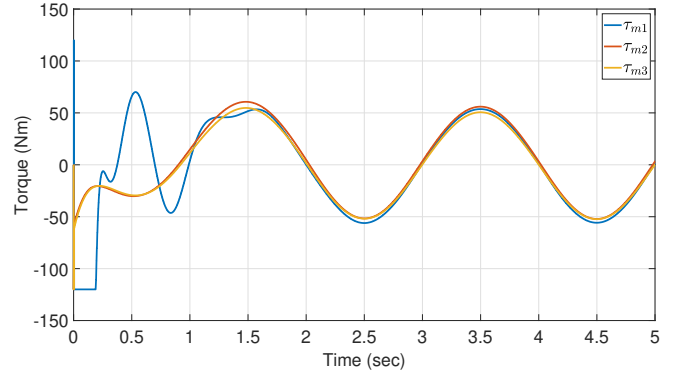
Based on the above analysis, the elastic joint should be placed closer to the end-effector to minimise fluctuations. In addition, the joint stiffness should be chosen appropriately. High joint stiffness helps reduce joint angular position error, but leads to high motor torque of the elastic joints, and vice versa.



(a) RRE



(b) RER



(c) ERR

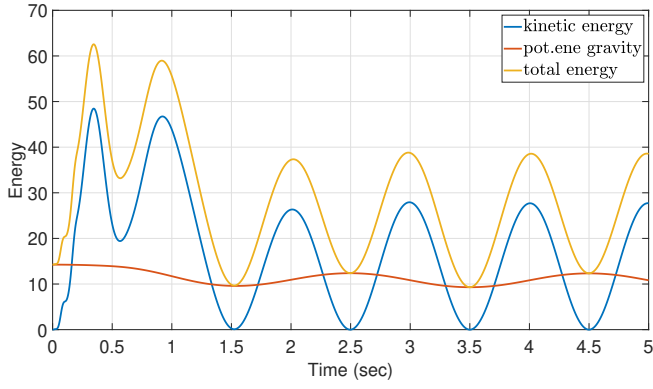
FIGURE 14: Motor torques in the three designs.

XIV. EXAMPLE

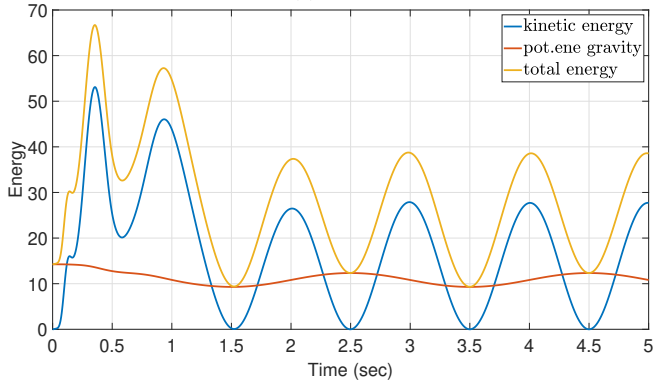
The dynamic vectors and matrices of a 3-DOF robotic manipulator are presented in this section as an example. The first two joints are rigid, and the last one is elastic. The dynamic model are calculated using Matlab. The kinematic diagram of the robot is shown in Figure 7, and the Denavit–Hartenberg table is shown in Table 2.

Motor 1 is attached to the base link, while motors 2 and 3 are mounted on link 1. The transformation matrices from the motor frames to the frames where they are mounted are:

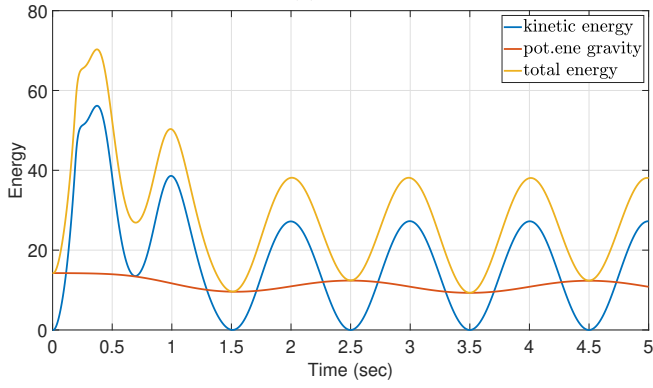
$$H_1^{M_1} = \begin{bmatrix} 0.7071 & 0.7071 & 0 & 0.0049 \\ -0.7071 & 0.7071 & 0 & -0.0049 \\ 0 & 0 & 1 & -0.0509 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(a) RRE



(b) RER



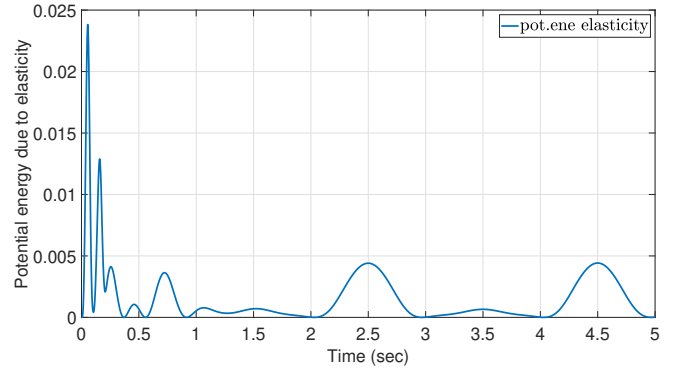
(c) ERR

FIGURE 15: Kinetic energy, potential energy due to gravity, and total energy.

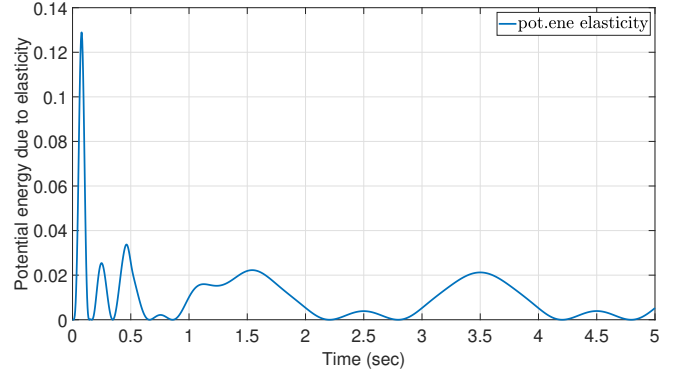
$$\mathbf{H}_{12}^{M_2} = \begin{bmatrix} \cos(q_1) & 0 & \sin(q_1) & 0 \\ \sin(q_1) & 0 & -\cos(q_1) & 0 \\ 0 & 1 & 0 & 0.053 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_{13}^{M_3} = \begin{bmatrix} \cos(q_1) & 0 & \sin(q_1) & 0 \\ \sin(q_1) & 0 & -\cos(q_1) & 0 \\ 0 & 1 & 0 & 0.053 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

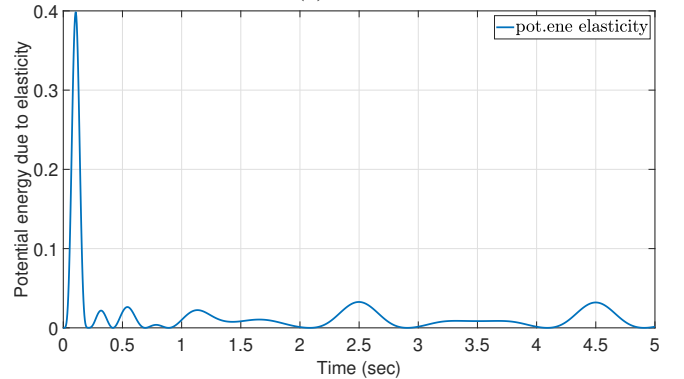
Positions of centres of mass of the joints, represented on the local frames, and the motors, represented on the frames



(a) RRE



(b) RER



(c) ERR

FIGURE 16: Potential energy due to elasticity.

where they are mounted, are $\mathbf{P}_{l_1}^1 = [0 \ 0 \ 0.053]^T$, $\mathbf{P}_{l_2}^2 = [0.1\cos(q_2) \ 0.1\sin(q_2) \ 0]^T$, $\mathbf{P}_{l_3}^3 = [-0.0155\cos(q_3) \ -0.0155\sin(q_3) \ 0]^T$, $\mathbf{P}_{m_1}^1 = [-0.0018 \ -0.0018 \ -0.0485]^T$, $\mathbf{P}_{m_2}^1 = [0.0645\sin(q_1) \ -0.0645\cos(q_1) \ 0.053]^T$, $\mathbf{P}_{m_3}^1 = [-0.0645\sin(q_1) \ 0.0645\cos(q_1) \ 0.053]^T$.

TABLE 2: DH table

Link	a	α	d	v
1	l_0	0	0	0
2	l_1	q_1	0	$\pi/2$
3	0	q_2	l_2	0
end-effector	0	$q_3 + \pi/2$	l_3	$-\pi/2$

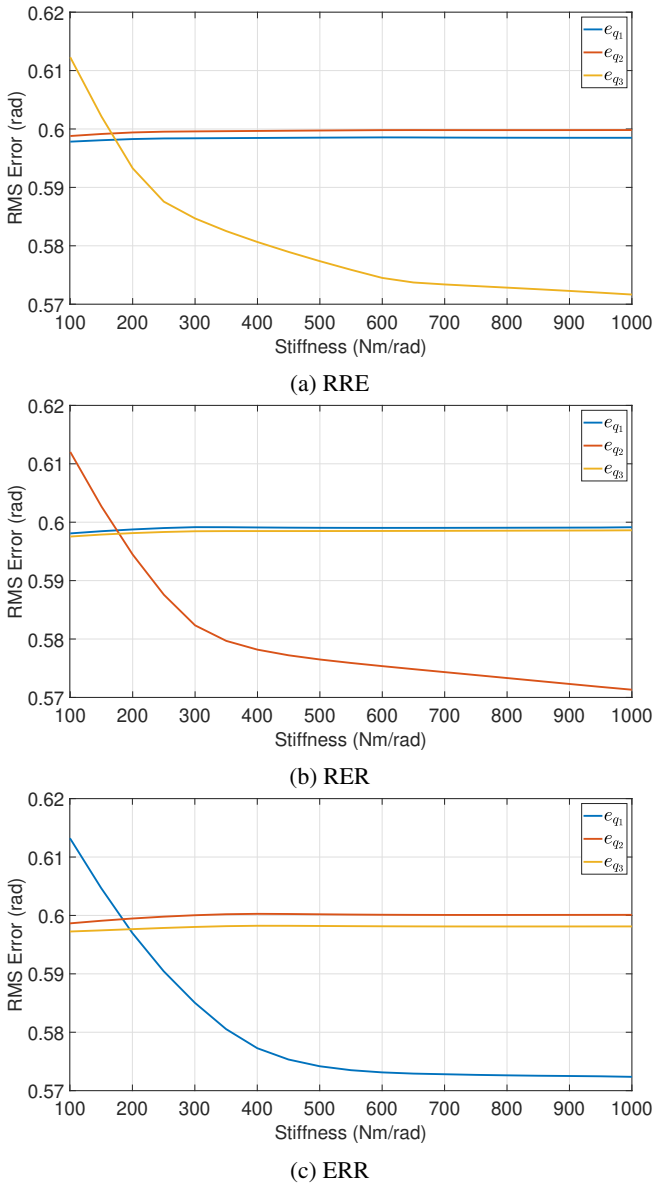


FIGURE 17: The plot shows the relationship between the root mean squares of the angular errors of the joints and the stiffness in different designs.

The elements of mass matrix $\mathbf{M}$ are:

$$\begin{aligned} M_{11} = & 0.15 \sin(2q_2) - 0.13 \sin(2q_2 + 2q_3) \\ & + 0.05 \cos(q_2 + q_3)^2 + 0.105 \cos(q_2)^2 \\ & - 0.006 \cos(q_2 + q_3) \cos(q_2) + 11.120 \end{aligned}$$

$$\begin{aligned} M_{12} = & 0.2 \sin(q_2 + q_3) - 0.15 \cos(q_2 + q_3) + 0.2 \cos(q_2) \\ & + 0.1 \sin(q_2) \end{aligned}$$

$$M_{13} = 0.2 \sin(q_2 + q_3) - 0.15 \cos(q_2 + q_3)$$

$$M_{22} = 10.605 - 0.006 \cos(q_3)$$

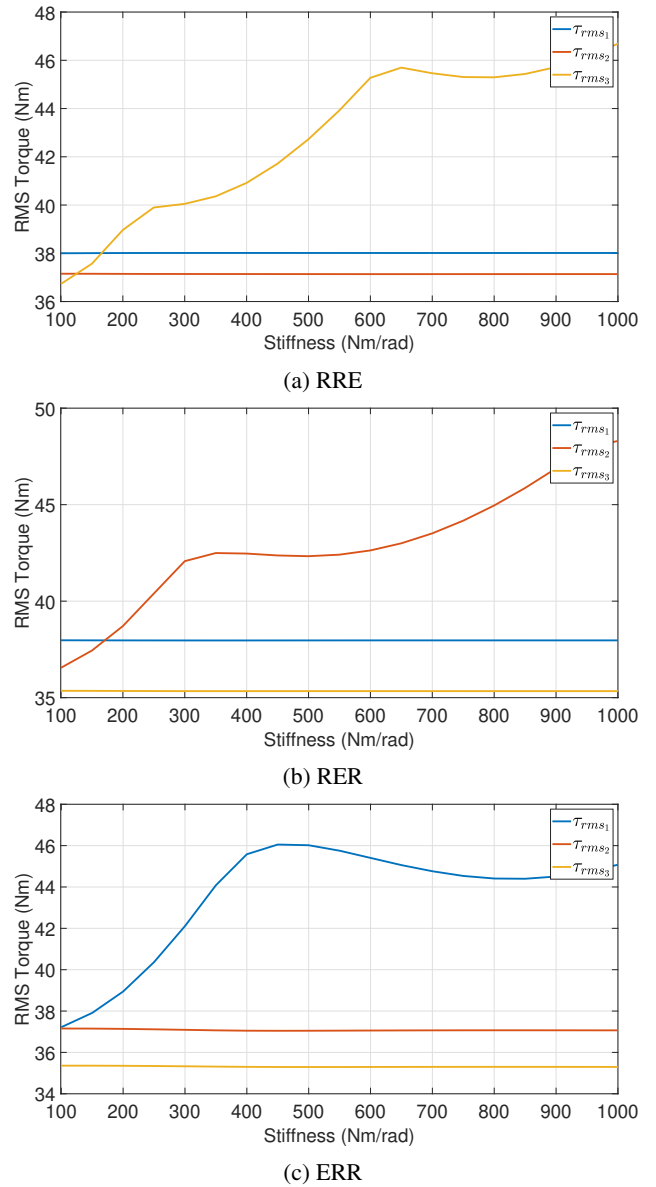


FIGURE 18: The plot shows the relationship between the root mean squares of the motor torques and the stiffness in different designs.

$$M_{23} = 0.25 - 0.003 \cos(q_3) \quad M_{33} = 0.25$$

The matrix $\mathbf{B}$ is a scalar in this case where $\mathbf{B} = 10$. The matrix $\mathbf{S}$ is a zero vector in this case, i.e $\mathbf{S} = [0 \ 0 \ 0]^T$. The elements of matrix $\mathbf{C}$ are:

$$\begin{aligned} C_{11} = & 0.5\dot{q}_3(0.0031 \sin(2q_2 + q_3) - 0.26 \cos(2q_2 + 2q_3) \\ & - 0.05 \sin(2q_2 + 2q_3) + 0.0031 \sin(q_3)) \\ & - 0.5\dot{q}_2(0.105 \sin(2q_2) - 0.3 \cos(2q_2) - 0.006 \sin(2q_2 + q_3) \\ & + 0.26 \cos(2q_2 + 2q_3) + 0.05 \sin(2q_2 + 2q_3)) \end{aligned}$$

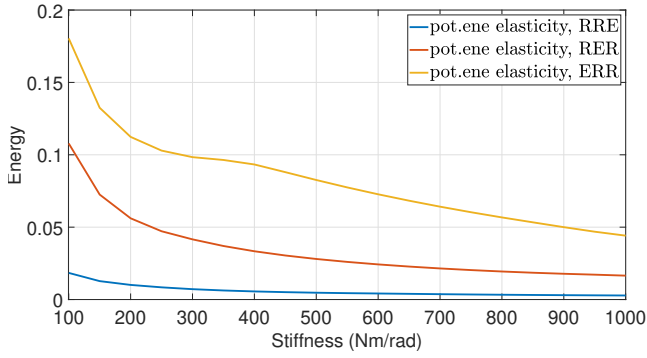


FIGURE 19: The plot shows the relationship between the root mean squares of the potential energy due to elasticity and the stiffness in different designs.

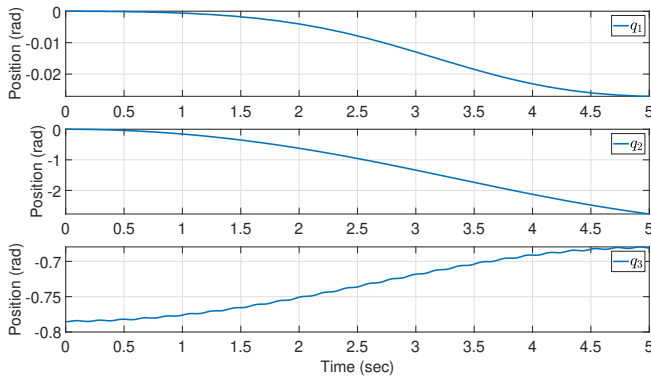


FIGURE 20: Joint positions of the robot.

$$\begin{aligned} C_{12} = & 0.15\dot{q}_1 \cos(2q_2) - 0.053\dot{q}_1 \sin(2q_2) \\ & - 0.13\dot{q}_1 \cos(2q_2 + 2q_3) - 0.025\dot{q}_1 \sin(2q_2 + 2q_3) \\ & + 0.2\dot{q}_2 \cos(q_2 + q_3) + 0.2\dot{q}_3 \cos(q_2 + q_3) \\ & + 0.15\dot{q}_2 \sin(q_2 + q_3) + 0.15\dot{q}_3 \sin(q_2 + q_3) + 0.1\dot{q}_2 \cos(q_2) \\ & - 0.2\dot{q}_2 \sin(q_2) + 0.0031\dot{q}_1 \sin(2q_2 + q_3) \end{aligned}$$

$$\begin{aligned} C_{13} = & 0.2\dot{q}_2 \cos(q_2 + q_3) - 0.025\dot{q}_1 \sin(2q_2 + 2q_3) \\ & - 0.13\dot{q}_1 \cos(2q_2 + 2q_3) + 0.2\dot{q}_3 \cos(q_2 + q_3) \\ & + 0.15\dot{q}_2 \sin(q_2 + q_3) + 0.15\dot{q}_3 \sin(q_2 + q_3) \\ & + 0.002\dot{q}_1 \sin(q_3) + 0.002\dot{q}_1 \sin(2q_2 + q_3) \end{aligned}$$

$$\begin{aligned} C_{21} = & 0.02\dot{q}_1(2.625 \sin(2q_2) - 7.5 \cos(2q_2) \\ & - 0.155 \sin(2q_2 + q_3) + 6.5 \cos(2q_2 + 2q_3) \\ & + 1.256 \sin(2q_2 + 2q_3)) \end{aligned}$$

$$C_{22} = 0.0031\dot{q}_3 \sin(q_3), \quad C_{23} = 0.0031 \sin(q_3)(\dot{q}_2 + \dot{q}_3)$$

$$\begin{aligned} C_{31} = & -0.1\dot{q}_1(0.016 \sin(2q_2 + q_3) - 1.3 \cos(2q_2 + 2q_3) \\ & - 0.251 \sin(2q_2 + 2q_3) + 0.016 \sin(q_3)) \end{aligned}$$

$$C_{32} = -0.0031 * \dot{q}_2 \sin(q_3), \quad C_{33} = 0$$

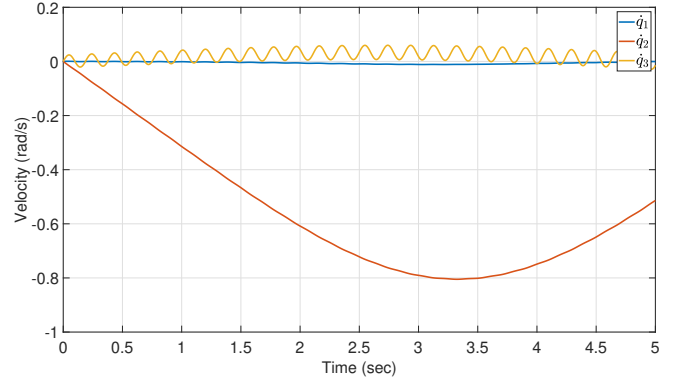


FIGURE 21: Joint velocities of the robot.

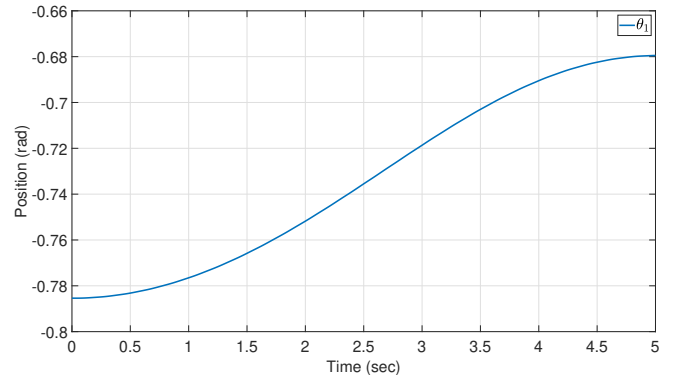


FIGURE 22: Motor position of the robot's elastic joint.

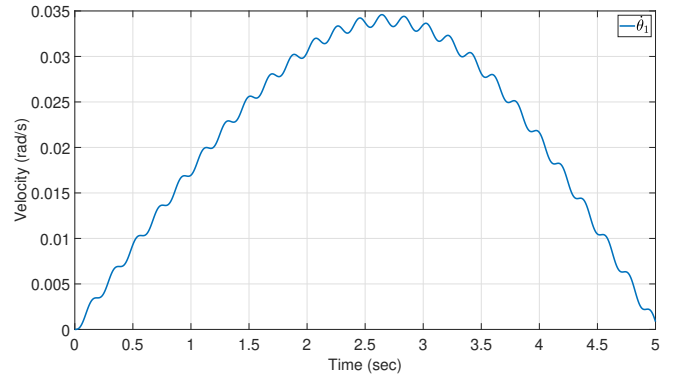


FIGURE 23: Motor velocity of the robot's elastic joint.

The matrices C_1 and C_2 are zero vectors in this case, i.e. $C_1 = C_2^T = [0 \ 0 \ 0]^T$. The gravitational vector G is defined as:

$$G = \begin{bmatrix} 0 \\ 3.434 \cos(q_2) - 0.152 \cos(q_2 + q_3) \\ -0.152 \cos(q_2 + q_3) \end{bmatrix}$$

XV. SIMULATIONS

In this section, simulations are performed under the conditions of zero input torque from the motors and the absence of friction, to validate the modelling method. The total energy is then computed to determine whether it is conserved. The simulated robot is the same as the one described in Section XIV.

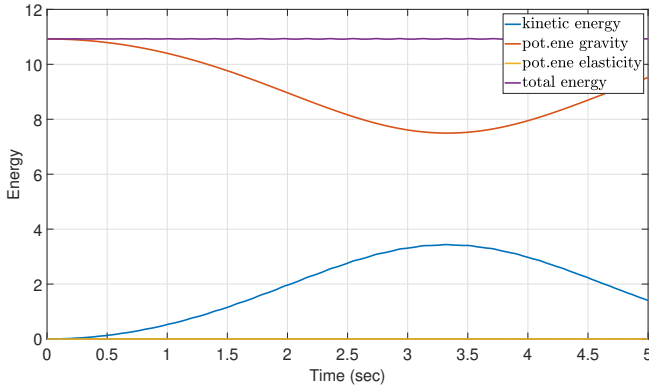


FIGURE 24: Energies of the robot, including kinetic energy, potential energy due to gravity, potential energy due to elasticity, and total energy.

The stiffness of the spring is 250 (Nm/rad). The simulation is run within 5 seconds.

Figures 20 and 21 show the joint positions and velocities, respectively. Figures 22 and 23 show the motor positions and velocities, respectively. In joint 3, the velocity oscillates due to the joint elasticity. The energies of the robot are shown in Figure 24. The total energy is conserved. Therefore, the modelling algorithm is verified.

XVI. CONCLUSION

Using rigid and elastic joints in a robotic manipulator configuration can limit drawbacks and enhance the benefits of these types of joints. Despite being introduced in previous work, to the best of the authors' knowledge, an inclusive and systematic article for the mathematical modelling of robotic manipulators with mixed rigid-elastic joints is still lacking. This paper provides a detailed, accessible, and comprehensive article on the modelling process using different methods of robotic manipulators with mixed joint types. First of all, the Lagrangian method is used to model robots with mixed rigid-elastic joints. The constraint on motor positions is relaxed. The unified rigid bodies are introduced. Furthermore, the skew-symmetric property and linear expression of the dynamic model with respect to the dynamic parameters are presented. The representation of the dynamic matrices and vectors with respect to the dynamic parameters is demonstrated. Secondly, the Newton-Euler method is adopted to derive the model of robots with mixed rigid-elastic joints. An inverse dynamic algorithm is derived, where the forces/torques are calculated from the provided accelerations, velocities, and positions of the joints. Based on this algorithm, a forward dynamic algorithm is proposed, where the accelerations, velocities, and positions of the joints are calculated from the provided forces/torques. In addition, the impact of different joint stiffness and the elastic joint placement is analysed. Finally, an example and simulation are given to illustrate the modelling process of robotic manipulators with mixed rigid-elastic joints.

APPENDIX A IMPORTANT RESULTS

A. MATHEMATICAL MODELS

The mathematical model of robotic manipulators with all elastic joints is:

$$\begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} \\ \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{K}\boldsymbol{\eta}_l(\boldsymbol{\eta}_l\mathbf{q} - \boldsymbol{\theta}) \\ \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{\eta}_l\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_m + \boldsymbol{\tau}_{fr,m} \end{bmatrix}$$

where $\mathbf{q}$ is the $N \times 1$ vector of joint variables, $\boldsymbol{\theta}$ is the $N \times 1$ vector of positions of motors of elastic joints, $\mathbf{M}(\mathbf{q})$ is the $N \times N$ mass matrix, $\mathbf{S}(\mathbf{q})$ is the $N \times N$ coupling matrix, $\mathbf{B}$ is the $N \times N$ inertia matrix of the motors of elastic joints, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})$ are $N \times N$ Coriolis and centrifugal matrices, $\mathbf{G}(\mathbf{q})$ is the $N \times 1$ gravitational vector, $\mathbf{K}$ is the stiffness matrix, $\boldsymbol{\eta}_l$ is the matrix containing torque transmission ratios, $\boldsymbol{\tau}_{ex}$ is the vector of external torques affecting the joints, $\boldsymbol{\tau}_{fr,l}$ is the vector of friction torques of the joints, $\boldsymbol{\tau}_{fr,m}$ is the vector of friction torques of the motors of elastic joints, and $\boldsymbol{\tau}_m$ is the vector of motor torques.

The mathematical model of robotic manipulators with mixed rigid-elastic joints is:

$$\begin{bmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{S}(\mathbf{q}) \\ \mathbf{S}(\mathbf{q})^T & \mathbf{B} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} \\ \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\tau}_{elas} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_q + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \boldsymbol{\tau}_m + \boldsymbol{\tau}_{fr,m} \end{bmatrix}$$

where $\mathbf{q}$ is the $N \times 1$ vector of joint variables, $\boldsymbol{\theta}$ is the $N_e \times 1$ vector of positions of motors of elastic joints, $\mathbf{M}(\mathbf{q})$ is the $N \times N$ mass matrix, $\mathbf{S}(\mathbf{q})$ is the $N \times N_e$ coupling matrix, $\mathbf{B}$ is the $N_e \times N_e$ inertia matrix of the motors of elastic joints, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})$ are $N \times N$, $N \times N_e$, $N_e \times N$ Coriolis and centrifugal matrices, respectively, $\mathbf{G}(\mathbf{q})$ is the $N \times 1$ gravitational vector, $\boldsymbol{\tau}_{elas}$ is the vector of elastic torques, $\boldsymbol{\tau}_{ex}$ is the vector of external torques affecting the joints, $\boldsymbol{\tau}_{fr,l}$ is the vector of friction torques of the joints, $\boldsymbol{\tau}_{fr,m}$ is the vector of friction torques of the motors of elastic joints, $\boldsymbol{\tau}_q$ is the vector of load side torques, and $\boldsymbol{\tau}_m$ is the vector of motor torques of elastic joints.

B. SKEW-SYMMETRY

If the element C_{ij} of matrix $\mathbf{C}$ is defined as:

$$C_{ij} = \frac{1}{2} \sum_{k=1}^N \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_k$$

then, matrix $\mathbf{W} = \dot{\mathbf{M}} - 2\mathbf{C}$ is skew-symmetric.

Define matrices $\mathcal{M}$ and $\mathcal{C}$ as:

$$\mathcal{M} = \begin{bmatrix} \mathbf{M} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{B} \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} \mathbf{C} & \mathbf{C}_1 \\ \mathbf{C}_2 & \mathbf{0} \end{bmatrix}$$

If the element C_{ij} of matrix $\mathcal{C}$ is calculated as:

$$C_{ij} = \frac{1}{2} \sum_{k=1}^{N+N_e} \left(\frac{\partial \mathcal{M}_{ij}}{\partial \Theta_k} + \frac{\partial \mathcal{M}_{ik}}{\partial \Theta_j} - \frac{\partial \mathcal{M}_{jk}}{\partial \Theta_i} \right) \dot{\Theta}_k$$

then, the matrix $\mathcal{W} = \mathcal{M} - 2\mathcal{C}$ is skew-symmetric.

C. LINEARITY OF THE DYNAMIC MATHEMATICAL MODEL WITH RESPECT TO THE DYNAMIC PARAMETERS

The dynamic equations of robotic manipulators with mixed rigid-elastic joints can be written as:

$$\begin{cases} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{S}(\mathbf{q})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\mathbf{q}) \\ \quad = \mathbf{Y}_q \boldsymbol{\pi}_q = \boldsymbol{\tau}_q + \boldsymbol{\tau}_{ex} + \boldsymbol{\tau}_{fr,l} \\ \mathbf{S}(\mathbf{q})^T \ddot{\mathbf{q}} + \mathbf{B}\ddot{\boldsymbol{\theta}} + \mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \boldsymbol{\tau}_{elas} \\ \quad = \mathbf{Y}_\theta \boldsymbol{\pi}_\theta + \boldsymbol{\tau}_{elas} = \boldsymbol{\tau}_{me} + \boldsymbol{\tau}_{fr,m} \end{cases}$$

where $\mathbf{Y}_q$, $\boldsymbol{\pi}_q$, $\mathbf{Y}_\theta$, and $\boldsymbol{\pi}_\theta$ are calculated as in Equations (82) and (85).

D. REPRESENT DYNAMIC MATRICES AND VECTORS WITH RESPECT TO DYNAMIC PARAMETERS

The mass matrix can be decomposed into three matrices:

$$\mathbf{M}(\mathbf{q}) = \mathbf{M}_J(\mathbf{q}) + \mathbf{M}_{rmc}(\mathbf{q}) + \mathbf{M}_{rmi}$$

where the matrix $\mathbf{M}_J(\mathbf{q})$ is defined as:

$$\mathbf{M}_J(\mathbf{q}) = \sum_{i=1}^N \left(m_i (\mathbf{J}_{vO_i}^{i+1})^T \mathbf{J}_{vO_i}^{i+1} - (\mathbf{J}_{vO_i}^{i+1})^T [m_i \mathbf{r}_{o_i c_i}^{i+1}] \times \mathbf{J}_{\omega O_i}^{i+1} \right. \\ \left. - (\mathbf{J}_{\omega O_i}^{i+1})^T ([m_i \mathbf{r}_{o_i c_i}^{i+1}] \times) \mathbf{J}_{vO_i}^{i+1} + (\mathbf{J}_{\omega O_i}^{i+1})^T \bar{\mathbf{I}}_{o_i}^{i+1} \mathbf{J}_{\omega O_i}^{i+1} \right),$$

the matrix $\mathbf{M}_{rmc}(\mathbf{q}) = \mathbf{M}_c(\mathbf{q}) + \mathbf{M}_c(\mathbf{q})^T$, $\mathbf{M}_c(\mathbf{q}) = [\mathbf{M}_{c_1}(\mathbf{q}) \mathbf{M}_{c_2}(\mathbf{q}) \dots \mathbf{M}_{c_N}(\mathbf{q})]$, is an $N \times N$ symmetry matrix, and the columns of the matrix $\mathbf{M}_c(\mathbf{q})$ are calculated as:

$$\mathbf{M}_{c_i}(\mathbf{q}) = \begin{cases} 0 & \text{if joint } i \text{ is an elastic joint} \\ \eta_{m_i} \mathbf{I}_{m_{z_i}} (\mathbf{J}_{\omega O_{lm_i}}^{l_{m_i}+1})^T \mathbf{z}_{m_i}^{l_{m_i}+1} & \text{if joint } i \text{ is a rigid joint} \end{cases},$$

and the matrix $\mathbf{M}_{rmi} = \text{diag}(\mathbf{M}_{rmi_1}, \mathbf{M}_{rmi_2}, \dots, \mathbf{M}_{rmi_N})$ is a constant diagonal $N \times N$ matrix, where its diagonal elements are defined as:

$$\mathbf{M}_{rmi_i} = \begin{cases} 0 & \text{if joint } i \text{ is an elastic joint} \\ \eta_{m_i}^2 \mathbf{I}_{m_{z_i}} & \text{if joint } i \text{ is a rigid joint} \end{cases}.$$

The matrix $\mathbf{M}_J(\mathbf{q})$ collects the masses m_i , moments of inertia $m_i \mathbf{r}_{o_i c_i}^{i+1}$, and inertia tensors $\bar{\mathbf{I}}_{o_i}^{i+1}$ of the unified rigid bodies. It also includes the Jacobian matrices of the translational velocity and angular velocity of the joints, which show the relationship between these motions and the inertia of the robotic manipulator. Matrices $\mathbf{M}_{rmc}(\mathbf{q})$ and $\mathbf{M}_{rmi}$ collect the inertia of motors of the rigid joints around their axes of rotation. Matrix $\mathbf{M}_{rmc}(\mathbf{q})$ shows the coupling relationship between the motion of rigid joints motor and the motion of the other joints, while the matrix $\mathbf{M}_{rmi}$ shows the inertia of the motor's own motion. Notice that inertia matrices $\mathbf{M}_{rmc}(\mathbf{q})$ and $\mathbf{M}_{rmi}$ for the rigid joints respectively correspond to matrices $\mathbf{S}(\mathbf{q})$ and $\mathbf{B}$ for the elastic joints.

E. MODELLING ROBOTIC MANIPULATORS WITH MIXED RIGID-ELASTIC JOINTS USING RECURSIVE NEWTON-EULER ALGORITHM

The inverse dynamics algorithm to compute the joint forces/torques based on the given accelerations, velocities, and positions is summarised in Algorithm 1. This algorithm can be used in designing the control algorithm.

The forward dynamics algorithm to determine the joint accelerations, velocities, and positions from the applied forces/torques is summarised in Algorithm 2. This algorithm can be used to simulate the robotic manipulator.

APPENDIX B DYNAMICS CALCULATION PROGRAM

A dynamics calculation program using MATLAB is provided as supplementary materials with this work. Instructions for using the program are provided in this Appendix. The program consists of three groups of files: pre-calculation files, configuration files, and simulation files.

The pre-calculation files, named using the format "*precalculate_*.m*", include programs that calculate and store dynamic matrices and vectors in symbolic form, reducing the computational burden during simulations or empirical experiments. The file "*precalculate_dynamic_matrices_lagrange.m*" and "*precalculate_dynamic_matrices.m*" calculate the dynamic matrices and vectors using the Lagrangian method and the formulas from Section VIII, respectively.

The configuration files, named in the format "*config_*.m*", include the robot's dynamic and kinematic parameters, along with other configuration settings. The files named in the format "*config_limit_*.m*" include the limitation values of the robotic manipulator.

The simulation files include the output of the pre-calculation files, the "*controller.m*" file, the controller algorithm files, the main file, and the "*plotting.m*" file. The "*controller.m*" file can be used to switch between different control algorithms. The controller algorithm files, named in the format "*controller_*.m*", include the control algorithms for the robotic manipulator with mixed rigid-elastic joints. The main file, named "*robot_arm_sim_ode.m*", controls the flow of the simulation process. The "*plotting.m*" file is used to plot the simulation results.

The simulation steps are as follows:

- Step 1: Verify the dynamic and kinematic parameters as well as the limitation values in the configuration files.
- Step 2: Run the pre-calculation files to generate and store the symbolic form of the dynamic matrices and vectors. File "*precalculate_dynamic_matrices_lagrange.m*" should be run first, and its output will be used as a reference. Then file "*precalculate_dynamic_matrices.m*" is run to generate and store dynamic matrices and vectors for simulation or empirical experiments. The dynamic matrices and vectors from file "*precalculate_dynamic_matrices.m*" will be compared to those generated by file "*precalculate_dynamic_matrices_lagrange.m*". If the result does not match, the pre-calculation files should be checked.

- Step 3: Program the control algorithm into the "controller_*.m" file. Check that the control algorithm to be tested is set up correctly in file "controller.m".
- Step 4: Run the main file "robot_arm_sim_ode.m". Once the simulation is complete, run the "plotting.m" file to plot the results.

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